JOURNAL OF COMPUTATIONAL PHYSICS 52, 237–272 (1983)

On the Combination of Finite Element and Splitting-Up Methods in the Solution of Parabolic Equations

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Received July 21, 1982; revised March 8, 1983

A scheme combining finite element and splitting-up methods is suggested for the numerical solution of a parabolic equation in two dimensions. Approximation in space variables is implemented by the finite element method on a rectangular grid, triangulated by the diagonals. A finite-difference operator of the problem is split into four positive semidefinite one-dimensional operators acting along coordinate and diagonal directions. For the integration with respect to time, a two-cycle splitting-up scheme of the solution is used. The application of the method to a nonuniform grid topologically equivalent to a rectangular one is studied, and the stability conditions of the splitting-up method in this case are obtained.

INTRODUCTION

This study is natural continuation of the works devoted to the splitting-up methods [1–6] and their applications to the solution of evolutionary equations with the use of the Galerkin method [10–12].

A new splitting-up method is suggested for the difference operator of the finite element method, approximating the two-dimensional parabolic second-order equation. The equation has mixed derivatives of a specific form, natural for a wide class of problems in oceanology, atmospheric dynamics, and others. Solution of the equation is approximated on a triangulated domain by means of piecewise-linear functions. Splitting of the difference operator in this case is carried out not along two coordinate directions as ususal but along four coordinates including diagonal directions of the meshes. A similar algorithm was suggested in [21] in the case of finite difference approach on a regular grid.

Section 1 presents preliminary definitions and description of weak solutions. In Section 2, we obtain estimates for the continuous-in-time Galerkin method. Section 3 discusses properties of the difference operator as a result of application of the Galerkin method to the space operator of the problem. The splitting-up method for solving problems with respect to time is formulated and substantiated in Section 4. Section 5 illustrates results of the solution of the Galerkin equations by the splitting-up method on nonuniform grids. Section 6 presents a numerical example calculated by the method.

0021-9991/83 $3.00

1. Definitions

In a cylindrical domain $Q = \Omega \times (0, T]$ we consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} A \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} B \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} G \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} Z \frac{\partial u}{\partial x} = F, \tag{1.1}$$

where Ω is a bounded uniconnected domain of space R^2 with the boundary $S \in C^2$. Coefficients A, B, G, $Z \in C^2(\Omega)$ are the functions x, y; $F = F(x, y, t)$.

For Eq. (1.1) we set the initial conditions

$$u(x, y, 0) = 0, \tag{1.2}$$

and the two types of the boundary conditions,

$$u\,|_S = 0 \tag{1.3}$$

and

$$\left.\frac{\partial u}{\partial v}\right|_S = 0. \tag{1.4}$$

To solve the problem with boundary conditions (1.4), we additionally need to fulfill the relation $\int_\Omega F\,dx\,dy = 0$ which yields the fulfillment of the relation

$$\int_\Omega u\,dx\,dy = 0. \tag{1.4*}$$

In what follows we assume that solutions for the problem with boundary conditions (1.4) are selected from the class of functions satisfying (1.4*).

In (1.4) $\partial/\partial v$ is a conormal derivative to the curve S given by

$$\frac{\partial}{\partial v} = \cos(nx)A \frac{\partial}{\partial x} + \cos(ny)B \frac{\partial}{\partial y} + \cos(nx)G \frac{\partial}{\partial y} + \cos(ny)Z \frac{\partial}{\partial x},$$

where n is the external normal to the boundary S.

For the sake of simplicity, conditions (1.3), (1.4), (1.2) have been assumed to be homogeneous; however, this limitation is not basic.

We rewrite Eq. (1.1), dividing the symmetric and skew-symmetric parts of the differential operator with mixed derivatives, on the basis of the representation

$$C = (G - Z)/2, \qquad D = (G + Z)/2.$$

Then Eq. (1.1) becomes

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} A \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} B \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} D \frac{\partial u}{\partial y} \\ &\quad - \frac{\partial}{\partial y} D \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} C \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} C \frac{\partial u}{\partial x} = F. \end{aligned} \tag{1.5}$$

Equations of the form (1.1), (1.5) arise in oceanology, particularly, in calculations of distributions of different substances in seas and oceans [13]. In this case, the desired function u has a sense of the water impurity concentration, F is the impurity source intensity; the terms containing the coefficients A, B, D describe turbulent diffusion of the impurity in a water basin. The skew-symmetric terms of Eq. (1.5) describe the impurity advection by currents. Here the coefficient C has a sense of the stream function. For such problems, it is convenient to represent a skew-symmetric operator as a second-order operator since in this case the approximating finite-difference operator remains skew-symmetric and the conservation law of squared substance is carried out.

Introduce the notation

$$(u, v) = \int_\Omega uv\, d\Omega, \qquad \|u\|_{L_2(\Omega)} = (u, u)^{1/2},$$

$$\|u\|^2_{W_2^1(\Omega)} = \|u\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L_2(\Omega)},$$

$$\|u\|^2_{W_2^2(\Omega)} = \|u\|^2_{W_2^1(\Omega)} + \left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial y^2}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial^2 u}{\partial x\, \partial y}\right\|^2_{L_2(\Omega)},$$

$$\|u\|^2_{L_2(0,T;X)} = \int_0^T \|u\|^2_X\, dt, \qquad \|u\|_{L_\infty(0,T;X)} = \sup_{t\in[0,T]} \|u\|_X,$$

$$\|u\|^2_{W_2^{2,1}(Q)} = \|u\|^2_{L(0,T;W_2^2(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L(0,T;L_2(\Omega))}.$$

Here $W_2^p(\Omega)$ is the Sobolev space [14], $\dot{W}_2(\Omega)$ is the subspace formed by the closing in $W_2^1(\Omega)$ of the functions $C_0^\infty(\Omega)$ which are infinitely differentiable and equal to zero on S; $W_2^{2,1}(Q)$ is the Sobolev–Slobodetsky space [15]. If we assume that the conditions

$$\begin{gathered} A, B, D, C \in C^2(\Omega), \qquad F \in C^1(\Omega) \times C^0(0, T], \qquad A, B \geqslant \gamma_2, \\ 0 < \gamma_0(\xi_1^2 + \xi_2^2) \leqslant A\xi_1^2 + B\xi_2^2 + 2D\xi_1\xi_2 \leqslant \gamma_1(\xi_1^2 + \xi_2^2) < \infty, \\ A, B\, |D|, |C| \leqslant \gamma_3, \qquad \gamma_1, \gamma_2, \gamma_3 > 0, \end{gathered} \tag{1.6}$$

are fulfilled for the coefficients of Eq. (1.5), then there exists a norm generated by a bilinear form that corresponds to the operator of the problem

$$\|u\|_e^2 = I(u, u),$$

$$I(u, v) = \int_\Omega \left(A \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + B\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + D\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + D\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + C\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - C\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) dx\, dy.$$

The norm is equivalent to $\|\cdot\|_{W_2^1(\Omega)}$. In other words,

$$\gamma_4 \|u\|_{W_2^1(\Omega)} < \|u\|_e \leqslant \gamma_5 \|u\|_{W_2^1(\Omega)}, \qquad \gamma_4, \gamma_5 > 0. \tag{1.6*}$$

Here u belongs to $\mathring{W}_2^1(\Omega)$ or $W_2^1(\Omega)$ provided that (1.4*) is fulfilled. It should be remembered that in what follows we will make use of the equivalence of these norms.

The existence and uniqueness of the solutions of problems (1.2), (1.3), (1.4), (1.5) are considered in [16, 17]. It follows from the theorems there that if the coefficients of Eq. (1.5) satisfy conditions (1.6), then there exists the unique solution $u \in W_2^{2,1}(Q)$ of problems (1.5), (1.2), (1.3) and (1.5), (1.3), (1.4) that satisfy the estimate

$$\|u\|_{W_2^{2,1}(Q)} \leqslant C \|F\|_{L_2(0,T;L_2(\Omega))}.$$

For convenience we will introduce a weak form in space variables of the problems under consideration. The weak solution of problem (1.2), (1.3), (1.5) will be the function $\tilde{u} \in L_2(0, T; \mathring{W}_2^1(\Omega))$ such that

$$\frac{\partial \tilde{u}}{\partial t} \in L_2(0, T; \mathring{W}_2^1(\Omega)),$$

$$\left(\frac{\partial \tilde{u}}{\partial t}, v\right) + I(\tilde{u}, v) = (F, v), \qquad v \in \mathring{W}_2^1(\Omega), \quad t \in (0, T], \tag{1.7}$$

$$(\tilde{u}(x, y, 0), v) = 0.$$

We also define the weak solution of problem (1.4), (1.2), (1.5) as the function $\hat{u} \in L_2(0, T; W_2^1(\Omega))$ such that

$$\frac{\partial \hat{u}}{\partial t} \in L_2(0, T; W_2^1(\Omega)),$$

$$\left(\frac{\partial \hat{u}}{\partial t}, v\right) + I(\hat{u}, v) = (F, v), \qquad v \in W_2^1(\Omega), \quad t \in (0, T], \tag{1.8}$$

$$(\hat{u}(x, y, 0), v) = 0.$$

For an approximate solution of problems (1.7), (1.8), we cover the domain Ω with a rectangular mesh whose nodes are formed by the intersection of $x_i = x_0 + h \cdot i$, $y_j = y_0 + r \cdot j$. Here h, r are the numerical parameters denoting the distance between the lines, and $O(h/r) = O(1)$. Rectangles of the mesh are triangulated with the diagonals depending on the sign of the value

$$J_{i+1,j+1}^{i,j} = \int_{\delta_{i+1j+1}^{i,j}} D \, dx \, dy,$$

$$\delta_{i+1,j+1}^{i,j} = \{(x, y): (x, y) \in \bar{\Omega}, x_i \leqslant x \leqslant x_{i+1}, y_j \leqslant y \leqslant y_{j+1}\}.$$

If $J_{i+1,j+1}^{i,j} \geqslant 0$ the cell $\delta_{i+1,j+1}^{i,j}$ is triangulated by a positively directed diagonal; if $J_{i+1,j+1}^{i,j} < 0$ it is triangulated by a negatively directed diagonal.

We define the domain $\hat{\Omega} \supset \Omega$ with the boundary $\hat{S}$ as the least combination of triangles T_k containing $\bar{\Omega}$ and denote a set of pairs of indices (i,j) of the nodes (x_i, y_j) belonging to $\hat{\Omega}$ by $\hat{R}^h$ and a set of indices (i,j) belonging to $\hat{S}$ by $\hat{\Gamma}^h$.

Let us consider also the domain $\tilde{\Omega} \subset \Omega$ with the boundary $\tilde{S}$ which is the largest combination of triangles belonging to Ω. We denote a set of indices (i,j) of the mesh nodes belonging to Ω by $\tilde{R}^h$ and a set of indices corresponding to the boundary nodes $\tilde{S}$ by $\tilde{\Gamma}^h$. In what follows we will assume that

$$\int_{T_k \cap \Omega} dx\, dy \geqslant shr, \qquad s = \text{const} > 0,$$

for any triangle $T_k \subset \hat{\Omega}$. We determine also the domains Ω_1, Ω_2 by the relations

$$\begin{aligned} \Omega_1 &= \left\{(x,y)\colon (x,y) \in \bigcup_{i,j} \delta^{i,j}_{i+1,j+1} \cap \Omega, J^{i,j}_{i+1,j+1} \geqslant 0\right\}, \\ \Omega_2 &= \left\{(x,y)\colon (x,y) \in \bigcup_{i,j} \delta^{i,j}_{i+1,j+1} \cap \Omega, J^{i,j}_{i+1,j+1} < 0\right\}. \end{aligned} \tag{1.9}$$

and denote the boundaries Ω_1, Ω_2 by S_1, S_2, respectively. Besides,

$$\begin{aligned} \tilde{R}^h_1 &= \{(i,j)\colon (i,j) \in \tilde{R}^h, (x_i, y_j) \in \Omega_1\}, \\ \tilde{R}^h_2 &= \{(i,j)\colon (i,j) \in \tilde{R}^h, (x_i, y_j) \in \Omega_2\}, \\ \hat{R}^h_1 &= \{(i,j)\colon (i,j) \in \hat{R}^h, (x_i, y_j) \in \Omega_1\}, \\ \hat{R}^h_2 &= \{(i,j)\colon (i,j) \in \hat{R}^h, (x_i, y_j) \in \Omega_2\}. \end{aligned}$$

Let $\tilde{\Gamma}^h_1$, $\tilde{\Gamma}^h_2$, $\hat{\Gamma}^h_1$, $\hat{\Gamma}^h_2$ be the corresponding sets of indices of the boundary points.

For every node $(i,j) \in \hat{R}^h \cup \hat{\Gamma}^h$ we determine the function $\omega_{i,j}(x,y)$ with the compact support $\kappa_{i,j}$ which is continuous in Ω, linear on each triangle of the domain $\hat{\Omega} \cup \hat{S}$, such that

$$\begin{aligned} \omega_{i,j}(x_m, y_n) &= 1, \qquad (i,j) = (m,n), \\ &= 0, \qquad (i,j) \neq (m,n), \end{aligned} \qquad (i,j), (m,n) \in \hat{R}^h \cup \hat{\Gamma}^h. \tag{1.10}$$

Further let $\Phi = \{\varphi_{i,j}(t)\}$ be a difference function defined in the nodes (x_i, y_j). Consider the functions

$$\tilde{\varphi}(x,y,t) = \sum_{(i,j) \in \tilde{R}^h} \varphi_{i,j}(t)\, \omega_{i,j}(x,y), \tag{1.11}$$

$$\hat{\varphi}(x,y,t) = \sum_{(i,j) \in \hat{R}^h \cup \hat{\Gamma}^h} \varphi_{i,j}(t)\, \omega_{i,j}(x,y). \tag{1.12}$$

The sets of functions of form (1.11) and (1.12) will be denoted by $\dot{H}^h$ and H^h, respectively. Besides, let

$$\dot{M}^h = \text{Span}\{\omega_{m,n}\colon (m,n) \in \tilde{R}^h\} \subset \dot{W}^1_2(\Omega),$$

$$M^h = \text{Span}\{\omega_{m,n}\colon (m,n) \in \hat{R}^h \cup \hat{\Gamma}^h\} \subset W^1_2(\Omega).$$

Let the function $\tilde{\varphi}(x, y, t) \in \dot{H}^h$ satisfying integro-differential relation

$$\left(\frac{\partial \tilde{\varphi}}{\partial t}, v\right) + I(\tilde{\varphi}, v) = (F, v), \qquad v \in \dot{M}^h, \tag{1.13}$$

$$(\tilde{\varphi}(x, y, 0), v) = 0, \tag{1.14}$$

be an approximate solution of (1.7). We determine an approximate solution of problem (1.8) as the function $\hat{\varphi}(x, y, t) \in H^h$, satisfying the integro-differential relation

$$\left(\frac{\partial \tilde{\varphi}}{\partial t}, v\right) + I(\tilde{\varphi}, v) = (F, v), \qquad v \in M^h, \tag{1.15}$$

$$(\hat{\varphi}(x, y), 0), v) = 0. \tag{1.16}$$

2. Galerkin Method

Let us obtain estimates for the differences $\tilde{u} - \tilde{\varphi}$, $\hat{u} - \hat{\varphi}$ of solutions of problems (1.7), (1.13) and (1.8), (1.15), respectively, following [18].

Determine the functions $\tilde{U}$, $\hat{U}$ as projections of the $\tilde{u}$, $\hat{u}$ solutions onto subspaces $\dot{H}^h$,H^h:

$$\tilde{U} = \{\tilde{u}(x_i, y_j, t) \colon (i, j) \in \tilde{R}^h \cup \tilde{\Gamma}^h;\ \tilde{U}(x, y, t) \in \dot{H}^h\}, \tag{2.1}$$

$$\hat{U} = \{\hat{u}(x_i, y_j, t) \colon (i, j) \in \hat{R}^h \cup \hat{\Gamma}^h;\ \hat{U}(x, y, t) \in H^h\}. \tag{2.2}$$

Consider the differences $\tilde{u} - \tilde{U} = \tilde{\eta}$ and $\hat{u} - \hat{U} = \hat{\eta}$. They are estimated on the basis of the theorem of approximation 2.3.1* [8]. The estimates can be written in the form

$$\|\tilde{\eta}\|_{L_2(0,T;\dot{W}_2^1(\Omega))} \leqslant c h_0^{1/2} \|\tilde{u}\|_{L_2(0,T;W_2^2(\Omega))}, \qquad h_0 = \max(h, r), \tag{2.3}$$

$$\|\hat{\eta}\|_{L_2(0,T;W_2^1(\Omega))} \leqslant c h_0 \|\hat{u}\|_{L_2(0,T;W_2^2(\Omega))}. \tag{2.4}$$

We can write similar estimates for the derivatives,

$$\left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|_{L_2(0,T;\dot{W}_2^1(\Omega))} \leqslant c h_0^{1/2} \left\|\frac{\partial \tilde{u}}{\partial t}\right\|_{L_2(0,T;W_2^2(\Omega))}, \tag{2.5}$$

$$\left\|\frac{\partial \hat{\eta}}{\partial t}\right\|_{L_2(0,T;W_2^1(\Omega))} \leqslant c h_0 \left\|\frac{\partial \hat{u}}{\partial t}\right\|_{L_2(0,T;W_2^2(\Omega))} \tag{2.6}$$

Reducing the order of accuracy of the problem with the Dirichlet boundary conditions is a consequence of "transporting" boundary conditions from the boundary S to $\tilde{S}$. If the boundary S is such that the distance between S and $\tilde{S}$ is of order of h_0^2 estimates (2.3), (2.5) will be of order h_0 [8].

Let us estimate differences of solutions of problems (1.7), (1.13) and (1.8), (1.15), respectively, making use of (2.3)–(2.6).

We introduce the notation

$$\begin{aligned} \tilde{\zeta} &= \tilde{u} - \tilde{\varphi}, \qquad \tilde{\eta} = \tilde{u} - \tilde{V}, \qquad \tilde{\xi} = \tilde{V} - \tilde{\varphi}, \\ \hat{\zeta} &= \hat{u} - \hat{\varphi}, \qquad \hat{\eta} = \hat{u} - \hat{V}, \qquad \hat{\xi} = \hat{V} - \hat{\varphi}. \end{aligned} \tag{2.7}$$

Here $\tilde{V} \in \dot{H}^h$, $\hat{V} \in H^h$ are some arbitrary functions. Consider problems (1.7), (1.13). From (1.7) it follows that

$$\left(\frac{\partial \tilde{V}}{\partial t}, v\right) + I(\tilde{V}, v) = -\left(\frac{\partial \tilde{\eta}}{\partial t}, v\right) - I(\tilde{\eta}, v) + (F, v), \qquad v \in \dot{M}^h. \tag{2.8}$$

Subtracting (1.13) from (2.8) yields

$$\left(\frac{\partial \tilde{\xi}}{\partial t}, v\right) + I(\tilde{\xi}, v) = -\left(\frac{\partial \tilde{\eta}}{\partial t}, v\right) - I(\tilde{\eta}, v), \qquad v \in \dot{M}^h. \tag{2.9}$$

Selecting $v = \tilde{\xi}(\cdot, t) \in \dot{M}^h$ in (2.9) and employing (1.6) we arrive at the inequality

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\|\tilde{\xi}\|^2_{L_2(\Omega)} + c_1\|\tilde{\xi}\|^2_{\dot{W}_2^1(\Omega)} \\ &\quad \leqslant \frac{1}{2\varepsilon}\left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|^2_{L_2(\Omega)} + \frac{c_0}{\varepsilon}\|\nabla \tilde{\eta}\|^2_{L_2(\Omega)} + \frac{\varepsilon}{2}\|\tilde{\xi}\|^2_{L_2(\Omega)} + C_0\varepsilon\|\nabla\tilde{\xi}\|^2_{L_2(\Omega)} \end{aligned} \tag{2.10}$$

which can be rewritten

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\|\tilde{\xi}\|^2_{L_2(\Omega)} + (c_1 - c_2\varepsilon)\|\tilde{\xi}\|^2_{\dot{W}_2^1(\Omega)} \\ &\quad \leqslant C_3\left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|^2_{\dot{W}_2^1(\Omega)} + C_4\|\tilde{\eta}\|^2_{\dot{W}_2^1(\Omega)}. \end{aligned} \tag{2.11}$$

Selecting $\varepsilon = (c_1 + 1)/c_2$ and integrating it over τ from 0 to t we come to

$$\begin{aligned} &\|\tilde{\xi}(t)\|_{L_2(\Omega)} + \|\tilde{\xi}\|_{L_2(0,t;\dot{W}_2^1(\Omega))} \\ &\quad \leqslant c_5\left(\left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|_{L_2(0,t;\dot{W}_2^1(\Omega))} + \|\tilde{\eta}\|_{L_2(0,t;\dot{W}_2^1(\Omega))}\right). \end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned} &\|\tilde{\xi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\tilde{\xi}\|_{L_2(0,T;\dot{W}_2^1(\Omega))} \\ &\quad \leqslant c_6\left(\left\|\frac{\partial \tilde{\eta}}{\partial t}\right\|_{L_2(0,T;\dot{W}_2^1(\Omega))} + \|\tilde{\eta}\|_{L_2(0,T;\dot{W}_2^1(\Omega))}\right). \end{aligned} \tag{2.13}$$

From (2.13) we obtain the inequality

$$\|\tilde{\zeta}\|_{L_\infty(0,T;L_2(\Omega))} + \|\tilde{\zeta}\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} \leqslant c_7 \left(\left\| \frac{\partial \tilde{\eta}}{\partial t} \right\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} + \|\tilde{\eta}\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} \right). \tag{2.14}$$

In deriving (2.14), we made use of the inequality of the triangle and the estimate

$$\|\eta\|_{L_\infty(0,T;L_2(\Omega))} \leqslant c \left(\|\eta\|_{L_2(0,T;L_2(\Omega))} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L_2(0,T;L_2(\Omega))} \right).$$

In (2.14) we can choose $\tilde{V} = \tilde{U}$. Taking into account (2.3), (2.5) then yields

$$\begin{gathered} \|\tilde{\zeta}\|_{L_\infty(0,T;\mathring{W}_2^1(\Omega))} + \|\tilde{\zeta}\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} \leqslant c_8 h_0^{1/2}, \\ c_8 = c \left(\|\tilde{u}\|_{L_2(0,T;W_2^2(\Omega))} + \left\| \frac{\partial \tilde{u}}{\partial t} \right\|_{L_2(0,T;W_2^2(\Omega))} \right). \end{gathered} \tag{2.15}$$

The estimate for $\hat{\zeta} = \hat{u} - \hat{\varphi}$ is obtained in a similar way,

$$\|\hat{\zeta}\|_{L_\infty(0,T;L_2(\Omega))} + \|\hat{\zeta}\|_{L_2(0,T;W_2^1(\Omega))} \leqslant c_9 h_0. \tag{2.16}$$

Thus, we prove

THEOREM. *Let conditions* (1.6) *be fulfilled for the parameters of problems* (1.7), (1.8). *Then the approximate solutions given by* (1.13), (1.15) *deviate, respectively, from the accurate solutions in the values satisfying* (2.15), (2.16).

3. DIFFERENCE OPERATOR

Let us study in more detail the form of differential difference equations (1.13), (1.14) and carry out some transformations. First we note that the grid operator at the time derivative is calculated by means of the "lumping" method, preserving the order of estimates (2.15), (2.16) [20].

Considering the above and the fact that the sets of functions $\{\omega_{m,n}\}$, $(m,n) \in \tilde{R}^h$ and $\{\omega_{m,n}\}$, $(m,n) \in \hat{R}^h \cup \hat{\Gamma}^h$ are bases in $\mathring{M}^h$ and M^h, respectively, we can write relations (1.13), (1.15) as systems of ordinary differential equations:

$$\begin{gathered} \frac{\partial \varphi_{m,n}}{\partial t} \int_\Omega \omega_{m,n}\, dx\, dy + I(\tilde{\varphi}, \omega_{m,n}) = (F, \omega_{m,n}), \\ \tilde{\varphi} \in \mathring{H}^h, \qquad (m,n) \in \tilde{R}^h, \qquad \varphi_{m,n}(0) = 0; \end{gathered} \tag{3.1}$$

$$\begin{gathered} \frac{\partial \varphi_{m,n}}{\partial t} \int_\Omega \omega_{m,n}\, dx\, dy + I(\hat{\varphi}, \omega_{m,n}) = (F, \omega_{m,n}), \\ \hat{\varphi} \in H^h, \qquad (m,n) \in \hat{R}^h \cup \hat{\Gamma}^h, \qquad \varphi_{m,n}(0) = 0. \end{gathered} \tag{3.2}$$

For convenience, we can present the systems of equations (3.1), (3.2) in the operator form. To begin with, we introduce the notation. Let us assume that the difference function $\Phi(t)=\{\varphi_{m,n}(t)\}$ belongs to the class $\mathring{Q}^h$, if

$$\Phi=\{\varphi_{m,n}\colon (m,n)\in \tilde{R}^h\cup\tilde{\Gamma}^h;\ \varphi_{m,n}=0,\ (m,n)\in\tilde{\Gamma}^h\}$$

and to the class Q^h, if

$$\Phi=\{\varphi_{m,n}\colon (m,n)\in\hat{R}^h\cup\hat{\Gamma}^h\}.$$

Then the systems of differential equations (3.1), (3.2) are formally written as one equation

$$\theta\frac{\partial\Phi}{\partial t}+\Lambda\Phi=f. \tag{3.3}$$

In (3.1) $\Phi\in\mathring{Q}^h$ while in (3.2) $\Phi\in Q^h$. The operator θ in the mesh nodes (m,n) is of the form

$$[\theta\Phi]_{m,n}=\varphi_{m,n}\int_\Omega \omega_{m,n}\,dx\,dy.$$

For problem (3.1) $(m,n)\in\tilde{R}^h$; for (3.2) $(m,n)\in\hat{R}^h\cup\hat{\Gamma}^h$.

The operator Λ in the node (m,n) is given by

$$\begin{aligned}
[\Lambda\phi]^{m,n}&=\alpha^{m,n}_{m,n}\varphi_{m,n}-\alpha^{m,n}_{m+1,n}\varphi_{m+1,n}-\alpha^{m,n}_{m-1,n}\varphi_{m-1,n}-\alpha^{m,n}_{m,n+1}\varphi_{m,n+1}-\alpha^{m,n}_{m,n-1}\varphi_{m,n-1}\\
&\quad-\alpha^{m,n}_{m+1,n+1}\varphi_{m+1,n+1}-\alpha^{m,n}_{m-1,n-1}\varphi_{m-1,n-1}-\alpha^{m,n}_{m-1,n+1}\varphi_{m-1,n+1}\\
&\quad-\alpha^{m,n}_{m+1,n-1}\varphi_{m+1,n-1}+\beta^{m,n}_{m+1,n}\varphi_{m+1,n}+\beta^{m,n}_{m-1,n}\varphi_{m-1,n}+\beta^{m,n}_{m,n+1}\varphi_{m,n+1}\\
&\quad+\beta^{m,n}_{m,n-1}\varphi_{m,n-1}+\beta^{m,n}_{m+1,n+1}\varphi_{m+1,n+1}+\beta^{m,n}_{m-1,n-1}\varphi_{m-1,n-1}\\
&\quad+\beta^{m,n}_{m-1,n+1}\varphi_{m-1,n+1}+\beta^{m,n}_{m+1,n-1}\varphi_{m+1,n-1}.
\end{aligned}\tag{3.4}$$

The coefficients α, β are obtained by substituting representations (1.11), (1.12) for $\tilde{\varphi}$, $\hat{\varphi}$ in the bilinear form $I(\varphi,\omega_{m,n})$. They are of the form

$$\begin{aligned}
\alpha^{m,n}_{m,n}&=\int_{\Omega_1\cap\kappa_{m,n}}A\left(\frac{\partial\omega_{m,n}}{\partial x}\right)^2dx\,dy+\int_{\Omega_1\cap\kappa_{m,n}}B\left(\frac{\partial\omega_{m,n}}{\partial y}\right)^2dx\,dy\\
&\quad-\int_{\Omega_1\cap\kappa_{m,n}}\left[-2D\frac{\partial\omega_{m,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy+\int_{\Omega_2\cap\kappa_{m,n}}A\left(\frac{\partial\omega_{m,n}}{\partial x}\right)^2dx\,dy\\
&\quad+\int_{\Omega_2\cap\kappa_{m,n}}B\left(\frac{\partial\omega_{m,n}}{\partial y}\right)^2dx\,dy+\int_{\Omega_2\cap\kappa_{m,n}}2D\frac{\partial\omega_{m,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\,dx\,dy,
\end{aligned}\tag{3.5a}$$

$$\alpha^{m,n}_{m\pm1,n}=\int_{\Omega_1\cap\kappa_{m\pm1,n}\cap\kappa_{m,n}} A\left[-\frac{\partial\omega_{m\pm1,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial x}\right]dx\,dy$$
$$-\int_{\Omega_1\cap\kappa_{m\pm1,n}\cap\kappa_{m,n}} D\left[\frac{\partial\omega_{m\pm1,n}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}+\frac{\partial\omega_{m\pm1,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy$$
$$+\int_{\Omega_2\cap\kappa_{m\pm1,n}\cap\kappa_{m,n}} A\left[-\frac{\partial\omega_{m\pm1,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial x}\right]dx\,dy$$
$$+\int_{\Omega_2\cap\kappa_{m\pm1,n}\cap\kappa_{m,n}} D\left[-\frac{\partial\omega_{m\pm1,n}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m\pm1,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5b)$$

$$\alpha^{m,n}_{m,n\pm1}=\int_{\Omega_1\cap\kappa_{m,n\pm1}\cap\kappa_{m,n}} B\left[-\frac{\partial\omega_{m,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy$$
$$-\int_{\Omega_1\cap\kappa_{m,n\pm1}\cap\kappa_{m,n}} D\left[\frac{\partial\omega_{m,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}+\frac{\partial\omega_{m,n\pm1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy$$
$$+\int_{\Omega_2\cap\kappa_{m,n\pm1}\cap\kappa_{m,n}} B\left[-\frac{\partial\omega_{m,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy$$
$$+\int_{\Omega_2\cap\kappa_{m,n\pm1}\cap\kappa_{m,n}} D\left[-\frac{\partial\omega_{m,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m,n\pm1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5c)$$

$$\alpha^{m,n}_{m\pm1,n\pm1}=\int_{\Omega_1\cap\kappa_{m\pm1,n\pm1}\cap\kappa_{m,n}} D\left[-\frac{\partial\omega_{m\pm1,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m\pm1,n\pm1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5d)$$

$$\alpha^{m,n}_{m\pm1,n\mp1}=-\int_{\Omega_2\cap\kappa_{m\pm1,n\mp1}\cap\kappa_{m,n}} D\left[\frac{\partial\omega_{m\pm1,n\mp1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}+\frac{\partial\omega_{m\pm1,n\mp1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5e)$$

$$\beta^{m,n}_{m\pm1,n}=\int_{\Omega\cap\kappa_{m\pm1,n}\cap\kappa_{m,n}} C\left[\frac{\partial\omega_{m\pm1,n}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m\pm1,n}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5f)$$

$$\beta^{m,n}_{m,n\pm1}=\int_{\Omega\cap\kappa_{m,n\pm1}\cap\kappa_{m,n}} C\left[\frac{\partial\omega_{m,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m,n\pm1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5g)$$

$$\beta^{m,n}_{m\pm1,n\pm1}=\int_{\Omega\cap\kappa_{m\pm1,n\pm1}\cap\kappa_{m,n}} C\left[\frac{\partial\omega_{m\pm1,n\pm1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m\pm1,n\pm1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy, \quad (3.5h)$$

$$\beta^{m,n}_{m\pm1,n\mp1}=\int_{\Omega\cap\kappa_{m\pm1,n\mp1}\cap\kappa_{m,n}} C\left[\frac{\partial\omega_{m\pm1,n\mp1}}{\partial y}\frac{\partial\omega_{m,n}}{\partial x}-\frac{\partial\omega_{m\pm1,n\mp1}}{\partial x}\frac{\partial\omega_{m,n}}{\partial y}\right]dx\,dy. \quad (3.5i)$$

To determine A in formulas (3.4), (3.5), $(m,n)\in\tilde{R}^h$, $\Phi\in\mathring{Q}^h$ for problem (3.1), $(m,n)\in\hat{R}^h\cup\hat{\Gamma}^h$, $\Phi\in Q^h$ for problem (3.2). From (3.5) it follows that the coefficients α, β have the properties

$$\alpha_{m,n}^{m,n} = -(\alpha_{m+1,n}^{m,n} + \alpha_{m-1,n}^{m,n}) - (\alpha_{m,n+1}^{m,n} + \alpha_{m,n-1}^{m,n})$$
$$- (\alpha_{m+1,n+1}^{m,n} + \alpha_{m-1,n-1}^{m,n}) - (\alpha_{m+1,n-1}^{m,n} + \alpha_{m-1,n+1}^{m,n}),$$

$$\alpha_{m\pm1,n}^{m,n} = \alpha_{m,n}^{m\pm1,n}, \qquad \alpha_{m,n\pm1}^{m,n} = \alpha_{m,n}^{m,n\pm1},$$
$$\alpha_{m\pm1,n\pm1}^{m,n} = \alpha_{m,n}^{m\pm1,n\pm1}, \qquad \alpha_{m\pm1,n\mp1}^{m,n} = \alpha_{m,n}^{m\pm1,n\mp1}, \tag{3.6}$$

$$\beta_{m\pm1,n}^{m,n} = -\beta_{m,n}^{m\pm1,n}, \qquad \beta_{m,n\pm1}^{m,n} = -\beta_{m,n}^{m,n\pm1}$$
$$\beta_{m\pm1,n\pm1}^{m,n} = -\beta_{m,n}^{m\pm1,n\pm1}, \qquad \beta_{m\pm1,n\mp1}^{m,n} = -\beta_{m,n}^{m\pm1,n\mp1}. \tag{3.7}$$

Next we obtain expressions of α for the two main types of the compact support $\kappa_{m,n}$ of the function $\omega_{m,n}$, namely, when $\kappa_{m,n} \in \Omega_1$ and $\kappa_{m,n} \in \Omega_2$ and the nodes are not neighbouring upon the boundary nodes. Then the compact supports are of the form shown in Fig. 1.

Having computed derivatives of the functions $\omega_{i,j}$ in (3.5), we obtain formulas for α [19]:

$(x_m, y_n) \in \Omega_1$:

$$\alpha_{m\pm1,n}^{m,n} = \int_{\Omega_1 \cap \kappa_{m\pm1,n} \cap \kappa_{m,n}} \left[\frac{A}{h^2} - \frac{D}{hr}\right] dx\,dy,$$

$$\alpha_{m,n\pm1}^{m,n} = \int_{\Omega_1 \cap \kappa_{m,n\pm1} \cap \kappa_{m,n}} \left[\frac{B}{r^2} - \frac{D}{hr}\right] dx\,dy,$$

$$\alpha_{m\pm1,n\pm1}^{m,n} = \frac{1}{hr}\int_{\Omega_1 \cap \kappa_{m\pm1,n\pm1} \cap \kappa_{m,n}} D\,dx\,dy, \qquad \alpha_{m\pm1,n\mp1}^{m,n} = 0. \tag{3.8}$$

$(x_m, y_n) \in \Omega_2$:

$$\alpha_{m\pm1,n}^{m,n} = \int_{\Omega_2 \cap \kappa_{m\pm1,n} \cap \kappa_{m,n}} \left[\frac{A}{h^2} + \frac{D}{hr}\right] dx\,dy,$$

$$\alpha_{m,n\pm1}^{m,n} = \int_{\Omega_2 \cap \kappa_{m,n\pm1} \cap \kappa_{m,n}} \left[\frac{B}{r^2} + \frac{D}{hr}\right] dx\,dy,$$

$$\alpha_{m\pm1,n\pm1}^{m,n} = 0, \qquad \alpha_{m\pm1,n\mp1}^{m,n} = -\frac{1}{hr}\int_{\Omega_2 \cap \kappa_{m\pm1,n\mp1} \cap \kappa_{m,n}} D\,dx\,dy,$$

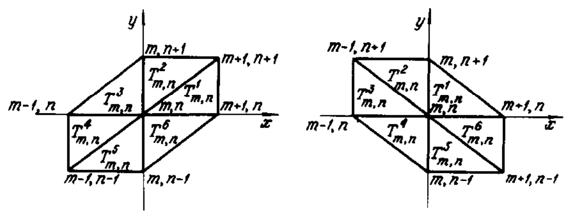

FIG. 1.

Note that in practical calculation of the integrals in (3.7), (3.8) for A, B, C, D, $\in C^2(\Omega)$ one can make use of the piecewise-linear representaion of the coefficients on the given triangulation Ω. Scheme (3.3) maintains the order of accuracy [8].

We consider now expression (3.4) for the operator Λ. It is not difficult to see that if we take into account the first relation of (3.6), the operator Λ can be written as a sum of four one-dimensional difference operators

$$\Lambda = \Lambda_{xx} + \Lambda_{yy} + \Lambda_{xy} + \Lambda_{yx} \tag{3.9}$$

of the form

$$\begin{aligned}
[\Lambda_{xx}\Phi]_{m,n} &= -\alpha^{m,n}_{m+1,n}\varphi_{m+1,n} + (\alpha^{m,n}_{m+1,n} + \alpha^{m,n}_{m-1,n})\,\varphi_{m,n} \\
&\quad - \alpha^{m,n}_{m-1,n}\varphi_{m-1,n} + \beta^{m,n}_{m+1,n}\varphi_{m+1,n} + \beta^{m,n}_{m-1,n}\varphi_{m-1,n}, \\
[\Lambda_{yy}\Phi]_{m,n} &= -\alpha^{m,n}_{m,n+1}\varphi_{m,n+1} + (\alpha^{m,n}_{m,n+1} + \alpha^{m,n}_{m,n-1})\,\varphi_{m,n} \\
&\quad - \alpha^{m,n}_{m,n-1}\varphi_{m,n-1} + \beta^{m,n}_{m,n+1}\varphi_{m,n+1} + \beta^{m,n}_{m,n-1}\varphi_{m,n-1}, \\
[\Lambda_{xy}\Phi]_{m,n} &= -\alpha^{m,n}_{m+1,n+1}\varphi_{m+1,n+1} + (\alpha^{m,n}_{m+1,n+1} + \alpha^{m,n}_{m-1,n-1})\,\varphi_{m,n} \\
&\quad - \alpha^{m,n}_{m-1,n-1}\varphi_{m-1,n-1} + \beta^{m,n}_{m+1,n+1}\varphi_{m+1,n+1} + \beta^{m,n}_{m-1,n-1}\varphi_{m-1,n-1}, \\
[\Lambda_{yx}\Phi]_{m,n} &= -\alpha^{m,n}_{m+1,n-1}\varphi_{m+1,n-1} + (\alpha^{m,n}_{m+1,n-1} + \alpha^{m,n}_{m-1,n+1})\,\varphi_{m,n} \\
&\quad - \alpha^{m,n}_{m-1,n+1}\varphi_{m-1,n+1} + \beta^{m,n}_{m+1,n-1}\varphi_{m+1,n-1} + \beta^{m,n}_{m-1,n+1}\varphi_{m-1,n+1}.
\end{aligned} \tag{3.10}$$

Let us investigate properties of the operators Λ_{xx}, Λ_{yy}, Λ_{yx}, Λ_{xy}. Introduce scalar products in grid spaces $\mathring{Q}^h$, Q^h,

$$\langle \Phi, \Psi \rangle_0 = \sum_{(i,j)\in \tilde{R}^h} \varphi_{i,j}\psi_{i,j}, \qquad \Phi, \Psi \in \mathring{Q}^h,$$

$$\langle \Phi, \Psi \rangle = \sum_{(i,j)\in \tilde{R}^h \cup \hat{\Gamma}^h} \varphi_{i,j}\psi_{i,j}, \qquad \Phi, \Psi \in Q^h.$$

Consider the functionals

$$\langle \Lambda_{xx}\Phi, \Phi \rangle_0, \quad \langle \Lambda_{xx}\Phi, \Phi \rangle, \quad \langle \Lambda_{yy}\Phi, \Phi \rangle_0, \quad \langle \Lambda_{yy}\Phi, \Phi \rangle,$$

$$\langle \Lambda_{xy}\Phi, \Phi \rangle_0, \quad \langle \Lambda_{xy}\Phi, \Phi \rangle, \quad \langle \Lambda_{yx}\Phi\Phi \rangle_0, \quad \langle \Lambda_{yx}\Phi, \Phi \rangle.$$

On the basis of (3.10)

$$\begin{aligned}
\langle \Lambda_{xx}\Phi, \Phi \rangle_0 = &\sum_{(m,n)\in \tilde{R}^h} (-\alpha^{m,n}_{m+1,n}\varphi_{m+1,n} + (\alpha^{m,n}_{m+1,n} + \alpha^{m,n}_{m-1,n})\,\varphi_{m,n} \\
&- \alpha^{m,n}_{m-1,n}\varphi_{m-1,n})\,\varphi_{m,n} \\
&+ \sum_{(m,n)\in \tilde{R}^h} (\beta^{m,n}_{m+1,n}\varphi_{m+1,n} + \beta^{m,n}_{m-1,n}\varphi_{m-1,n})\,\varphi_{m,n}.
\end{aligned} \tag{3.11}$$

If we use properties (3.7) in (3.11) the second sum equals zero. Making use of (3.6) for the first sum and rearranging the terms, we obtain

$$\begin{aligned}\langle \Lambda_{xx}\Phi, \Phi\rangle_0 = &\sum_{(m,n)\in \tilde{R}_1^h \cup \tilde{\Gamma}_1^x} \alpha_{m+1,n}^{m,n} (\varphi_{m,n} - \varphi_{m+1,n})^2 \\ &+ \sum_{(m,n)\in \tilde{R}_2^h \cup \tilde{\Gamma}_2^x} \alpha_{m+1,n}^{m,n} (\varphi_{m,n} - \varphi_{m+1,n})^2,\end{aligned}$$

where

$$\begin{aligned}\tilde{\Gamma}_1^x &= \{(m, n) \colon (m, n) \in \tilde{\Gamma}_1^h;\ (m-1, n)\ \bar{\in}\ \tilde{R}_1^h \cup \tilde{\Gamma}_1^h\},\\ \tilde{\Gamma}_2^x &= \{(m, n) \colon (m, n) \in \tilde{\Gamma}_2^h;\ (m-1, n)\ \bar{\in}\ \tilde{R}_2^h \cup \tilde{\Gamma}_2^h\}.\end{aligned}$$

From the latter, making use of (3.8), we have

$$\begin{aligned}\langle \Lambda_{xx}\Phi, \Phi\rangle_0 = &\sum_{(m,n)\in \tilde{R}_1^h \cup \tilde{\Gamma}_1^x} \int_{\kappa_{m\pm 1,n}\cap\kappa_{m,n}\cap\Omega_1} \left[\frac{A}{h^2} - \frac{D}{hr}\right] dx\, dy\, (\varphi_{m,n} - \varphi_{m+1,n})^2 \\ &+ \sum_{(m,n)\in \tilde{R}_2^h \cup \tilde{\Gamma}_2^x} \int_{\kappa_{m+1,n}\cap\kappa_{m,n}\cap\Omega_2} \left[\frac{A}{h^2} + \frac{D}{hr}\right] dx\, dy\, (\varphi_{m,n} - \varphi_{m+1,n})^2.\end{aligned} \tag{3.12}$$

Similarly we obtain relations for $\langle \Lambda_{xx}\Phi, \Phi\rangle$,

$$\begin{aligned}\langle \Lambda_{xx}\Phi, \Phi\rangle = &\sum_{(m,n)\in \hat{R}_1^h \cup \hat{\Gamma}_1^x} \int_{\kappa_{m+1,n}\cap\kappa_{m,n}\cap\Omega_1} \left[\frac{A}{h^2} - \frac{D}{hr}\right] dx\, dy\, (\varphi_{m,n} - \varphi_{m+1,n})^2 \\ &+ \sum_{(m,n)\in \hat{R}_2^h \cup \hat{\Gamma}_2^x} \int_{\kappa_{m+1,n}\cap\kappa_{m,n}\cap\Omega_2} \left[\frac{A}{h^2} + \frac{D}{hr}\right] dx\, dy\, (\varphi_{m,n} - \varphi_{m+1,n})^2,\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}\hat{\Gamma}_1^x &= \{(m, n) \colon (m, n) \in \hat{\Gamma}_1;\ (m-1, n)\ \bar{\in}\ \hat{R}_1^h\},\\ \hat{\Gamma}_2^x &= \{(m, n) \colon (m, n) \in \hat{\Gamma}_2;\ (m-1, n)\ \bar{\in}\ \hat{R}_2^h\}.\end{aligned}$$

From (3.12), (3.13) it follows that a sufficient condition to fulfill the inequalities

$$\langle \Lambda_{xx}\Phi, \Phi\rangle_0 \geqslant 0, \qquad \langle \Lambda_{xx}\Phi, \Phi\rangle \geqslant 0 \tag{3.14}$$

is the condition

$$\int_{T_{m,n}^k \cap \Omega} A\, dx\, dy \geqslant \frac{h}{r} \left| \int_{T_{m,n}^k \cap \Omega} D\, dx\, dy \right| \tag{3.15}$$

for any triangle $T_{m,n}^k \in \hat{\Omega}$. In a similar way we derive conditions of positive semidefiniteness of the difference operator Λ_{yy} on the classes of difference functions $\mathring{Q}^h$, Q^h:

$$\langle \Lambda_{yy}\Phi, \Phi\rangle_0 \geqslant 0, \qquad \langle \Lambda_{yy}\Phi, \Phi\rangle \geqslant 0. \tag{3.16}$$

The condition is of the form

$$\int_{T^k_{m,n}\cap\Omega} B\,dx\,dy \geqslant \frac{r}{h}\left|\int_{T^k_{m,n}\cap\Omega} D\,dx\,dy\right| \tag{3.17}$$

and should be fulfilled for any triangle $T^k_{m,n}\subset\hat{\Omega}$.

Let us consider the difference functional

$$\begin{aligned}\langle\Lambda_{xy}\Phi,\Phi\rangle_0 = &\sum_{(m,n)\in\tilde{R}^h}(-\alpha^{m,n}_{m+1,n+1}\varphi_{m+1,n+1}+(\alpha^{m,n}_{m+1,n+1}+\alpha^{m,n}_{m-1,n-1})\,\varphi_{m,n}\\ &-\alpha^{m,n}_{m-1,n-1}\varphi_{m-1,n-1})\,\varphi_{m,n}\\ &+\sum_{(m,n)\in\tilde{R}^h}(\beta^{m,n}_{m+1,n+1}\varphi_{m+1,n+1}+\beta^{m,n}_{m-1,n-1}\varphi_{m-1,n-1})\,\varphi_{m,n}.\end{aligned}$$

Using (3.6), (3.7) yields

$$\begin{aligned}&\langle\Lambda_{xy}\Phi,\Phi\rangle_0=\sum_{(m,n)\in\tilde{R}^h_1\cup\tilde{\Gamma}^{xy}_1}\alpha^{m,n}_{m+1,n+1}(\varphi_{m,n}-\varphi_{m+1,n+1})^2,\\ &\tilde{\Gamma}^{xy}_1=\{(m,n):(m,n)\in\Gamma^h_1;\,(m-1,n-1)\bar{\in}\tilde{R}^h_1\cup\tilde{\Gamma}^h_1\}.\end{aligned} \tag{3.18}$$

Here the fact that

$$\alpha^{m,n}_{m\pm1,n\pm1}=0 \qquad \text{if} \quad (\kappa_{m\pm1,n\pm1}\cap\kappa_{m,n})\subset\Omega_2$$

is taken into consideration. Substituting then the expressions for $\alpha^{m,n}_{m+1,n+1}$ from (3.8) we obtain

$$\begin{aligned}\langle\Lambda_{xy}\Phi,\Phi\rangle_0=&\sum_{(m,n)\in\tilde{R}^h_1\cup\tilde{\Gamma}^{xy}_1}\frac{1}{hr}\\ &\times\int_{\kappa_{m+1,n+1}\cap\kappa_{m,n}\cap\Omega_1} D\,dx\,dy\,(\varphi_{m,n}-\varphi_{m+1,n+1})^2.\end{aligned} \tag{3.19}$$

Since

$$\int_{\kappa_{m+1,n+1}\cap\kappa_{m,n}\cap\Omega_1} D\,dx\,dy\geqslant 0 \qquad \text{if} \quad (\kappa_{m+1,n+1}\cap\kappa_{m,n})\subset\Omega_1 \tag{3.20}$$

then from (3.19) it follows that

$$\langle\Lambda_{xy}\Phi,\Phi\rangle_0\geqslant 0. \tag{3.21}$$

Similarly,

$$\langle\Lambda_{xy}\Phi,\Phi\rangle\geqslant 0. \tag{3.22}$$

The functional $\langle \Lambda_{yx}\Phi, \Phi\rangle$ can be written

$$\langle \Lambda_{yx}\Phi, \Phi\rangle_0 = \sum_{(m,n)\in \tilde{R}_2^h \cup \tilde{\Gamma}_2^{xy}} \left(-\frac{1}{hr}\int_{\kappa_{m+1,n-1}\cap\kappa_{m,n}\cap\Omega_2} D\,dx\,dy\right) \times (\varphi_{m,n} - \varphi_{m+1,n-1})^2,$$

$$\tilde{\Gamma}_2^{yx} = \{(m,n): (m,n)\in\Gamma_2^h;\ (m-1,n+1)\,\bar{\in}\,\tilde{R}_2^h\cup\tilde{\Gamma}_2^h\}.$$

Since

$$\int_{\kappa_{m+1,n-1}\cap\kappa_{m,n}\cap\Omega_2} D\,dx\,dy < 0 \qquad \text{if} \quad (\kappa_{m+1,n-1}\cap\kappa_{m,n})\subset\Omega_2, \tag{3.23}$$

then

$$\langle \Lambda_{yx}\Phi, \Phi\rangle \geqslant 0. \tag{3.24}$$

In the same way we obtain the estimate for Λ_{yx} on the class of grid functions Q^h:

$$\langle \Lambda_{yx}\Phi, \Phi\rangle \geqslant 0. \tag{3.25}$$

It is of interest to obtain the upper bound for the operators Λ_{xx}, Λ_{yy}, Λ_{xy}, Λ_{yx}. It is easy to obtain the inequality

$$\langle \Lambda_{xx}\Phi, \Phi\rangle_0 \leqslant 4 \max_{(x,y)\in\Omega}\left(\frac{A}{h^2} - \frac{|D|}{hr}\right)\langle\Phi,\Phi\rangle_0\, hr,$$

for $\langle \Lambda_{xx}\Phi, \Phi\rangle_0$ if we apply relations (3.12) calculated above. Similarly

$$\begin{aligned}
&\langle \Lambda_{xx}\Phi, \Phi\rangle \leqslant 4 \max_{\Omega}\left(\frac{A}{h^2} - \frac{|D|}{hr}\right)\langle\Phi,\Phi\rangle\, hr,\\
&\langle \Lambda_{yy}\Phi, \Phi\rangle_0 \leqslant 4 \max_{\Omega}\left(\frac{B}{r^2} - \frac{|D|}{hr}\right)\langle\Phi,\Phi\rangle_0\, hr,\\
&\langle \Lambda_{yy}\Phi, \Phi\rangle \leqslant 4 \max_{\Omega}\left(\frac{B}{r^2} - \frac{|D|}{hr}\right)\langle\Phi,\Phi\rangle\, hr,\\
&\langle \Lambda_{xy}\Phi, \Phi\rangle_0 \leqslant 4 \max_{\Omega_1} D\langle\Phi,\Phi\rangle_0,\\
&\langle \Lambda_{xy}\Phi, \Phi\rangle \leqslant 4 \max_{\Omega_1} D\langle\Phi,\Phi\rangle,\\
&\langle \Lambda_{yx}\Phi, \Phi\rangle_0 \leqslant 4 \max_{\Omega_2} |D|\,\langle\Phi,\Phi\rangle_0,\\
&\langle \Lambda_{yx}\Phi, \Phi\rangle \leqslant 4 \max_{\Omega_2} |D|\,\langle\Phi,\Phi\rangle.
\end{aligned} \tag{3.26}$$

To conclude, let us prove the lower bound for the operator θ on classes of difference functions $\mathring{Q}^h$, Q^h. Consider the functional

$$\langle \theta\Phi, \Phi \rangle_0 = \sum_{(m,n)\in \tilde{R}^h} \varphi^2_{m,n} \int_{x_{m,n}} \omega_{m,n}\, dx\, dy \geqslant \tfrac{2}{3} rh\, \langle \Phi, \Phi \rangle_0. \tag{3.27}$$

In deriving (3.27), we made use of the fact that the minimum of triangles whose vertices can be the node $(m, n) \in \tilde{R}^h$ equals four.

The estimate for grid functions $\Phi \in Q^h$ depends on a minimal square of intersection of the triangles $T^k_{m,n} \subset \hat{\Omega}$ with the domain Ω. Since it is assumed to equal shr, then

$$\langle \theta\Phi, \Phi \rangle \geqslant shr \langle \Phi, \Phi \rangle. \tag{3.28}$$

Thus, summing up the results obtained in this section one can note that the differential equations (3.1), (3.2) can be presented in the operator form (3.3), where θ is a positive-definite grid operator satisfying estimates Λ (3.27), (3.28) on the classes of functions $\mathring{Q}^h$, Q^h, respectively, and the grid operators are fulfilled if conditions (3.15), (3.17), (3.20), (3.23) are presented as sum a sum of the four positive-definite operators Λ_{xx}, Λ_{yy}, Λ_{xy}, Λ_{yx} for which estimates (3.14), (3.16), (3.21), (3.22), (3.24), (3.25), (3.26) hold on the classes of difference functions $\mathring{Q}^h$ and Q^h.

Let us discuss in more detail conditions (3.15), (3.17). We will show that they are stronger than the ellipticity condition of the operator (1.6*) which can be rewritten in the form

$$AB - D^2 > 0 \tag{3.29}$$

or

$$A > (|D|/B)\,|D|, \tag{3.30}$$

$$B > (|D|/B)\,|D|. \tag{3.31}$$

Conditions (3.15), (3.17) signify that there exists such $p = \text{const}$ that

$$A \geqslant p\,|D|, \tag{3.32}$$

$$B \geqslant (1/p)\,|D|. \tag{3.33}$$

The question of existence of the number p requires that we consider several variants for the values of the coefficients A, B, D in the domain Ω:

1. $A > |D|,\quad B > |D|,$
2. $A > |D|,\quad B < |D|,\quad \forall (x, y) \in \Omega.$
3. $A < |D|,\quad B > |D|,$
4. $A \lesseqgtr |D|,\quad B \lesseqgtr |D|,$

Without going into details we will point to the final results:

1. If there is no additional data on A, B, D, conditions (3.32), (3.33) are fulfilled at least for $p=1$.
2. If $a=\max(A/|D|$, $b=\min(|D|/B)$, and $a \leqslant b$, then $a \leqslant p \leqslant b$.
3. If $a=\max(B/|D|)$, $b=\min(|D|/B)$, and $a \leqslant b$, then $a \leqslant p \leqslant b$.
4. The number p does not exist.

Hence, in order that a regular mesh exist, where one-dimensional operators are positive semi-definite, it is necessary for one-dimensional components of the differential operator to meet stronger conditions than the two-dimensional ellipticity condition.

4. Splitting-up Method

The results of Section 3 allow us to use the splitting-up methods for solving problems (3.1), (3.2) with respect to time. Particularly, we can apply the two-cycle splitting-up method treated in [5].

Thus let us consider

$$\theta \frac{\partial \Phi}{\partial t} + \Lambda \Phi = f, \qquad \Phi(0) = 0 \tag{4.1}$$

on the interval $(0, T]$. In this section we will not distinguish between problems (3.1) and (3.2), since the splitting-up method is formulated similarly for both problems.

The norm $\|\cdot\|_h$ used in this section is generated by the scalar product $\langle \cdot, \cdot \rangle_0$ or $\langle \cdot, \cdot \rangle$ depending on the type of the problem.

The operator θ in (4.1) is positive definite and estimate (3.27) or (3.28) is fulfilled for it. The operator Λ is representable in the form

$$\Lambda = \Lambda_{xx} + \Lambda_{yy} + \Lambda_{xy} + \Lambda_{yx}, \tag{4.2}$$

where

$$\Lambda_{xx} \geqslant 0, \qquad \Lambda_{yy} \geqslant 0, \qquad \Lambda_{xy} \geqslant 0, \qquad \Lambda_{yx} \geqslant 0.$$

Let us partition the interval $(0, T]$ into subintervals $t_j \leqslant t \leqslant t_{j+1}$ $(\tau = t_{j+1} - t_j)$. Following [5] we construct a system of difference equations consisting of a sequence of the Crank–Nicholson schemes for operators Λ_{xx}, Λ_{yy}, Λ_{xy}, Λ_{yx}, Λ_{yx}, Λ_{xy}, Λ_{yy}, Λ_{xx} on the interval $t_{j-1} \leqslant t \leqslant t_{j+1}$.

The scheme is of the form

$$
\begin{aligned}
(\theta + (\tau/2)\Lambda_{xx})\,\Phi_\tau^{j-(3/4)} &= (\theta - (\tau/2)\Lambda_{xx})\Phi_\tau^{j-1},\\
(\theta + (\tau/2)\Lambda_{yy})\,\Phi_\tau^{j-(1/2)} &= (\theta - (\tau/2)\Lambda_{yy})\,\Phi_\tau^{j-(3/4)},\\
(\theta + (\tau/2)\Lambda_{xy})\,\Phi_\tau^{j-(1/4)} &= (\theta - (\tau/2)\Lambda_{xy})\,\Phi_\tau^{j-(1/2)},\\
(\theta + (\tau/2)\Lambda_{yx})(\Phi_\tau^{j} - \tau f^j) &= (\theta - (\tau/2)\Lambda_{yx})\,\Phi_\tau^{j-(1/4)},\\
(\theta + (\tau/2)\Lambda_{yx})\,\Phi_\tau^{j+(1/4)} &= (\theta - (\tau/2)\Lambda_{yx})(\Phi_\tau^{j} - \tau f^j),\\
(\theta + (\tau/2)\Lambda_{xy})\,\Phi_\tau^{j+(1/2)} &= (\theta - (\tau/2)\Lambda_{xy})\,\Phi_\tau^{j+(1/4)},\\
(\theta + (\tau/2)\Lambda_{yy})\,\Phi_\tau^{j+(3/4)} &= (\theta - (\tau/2)\Lambda_{yy})\,\Phi_\tau^{j+(1/2)},\\
(\theta + (\tau/2)\Lambda_{xx})\,\Phi_\tau^{j+1} &= (\theta - (\tau/2)\Lambda_{xx})\,\Phi_\tau^{j+(3/4)},
\end{aligned}
\tag{4.3}
$$

where $f^j = f(t_j)$.

Excluding the functions $\Phi_\tau^{j+(\alpha/4)}$ ($\alpha = -3, -2, -1, 0, 1, 2, 3$), from (4.3), we can rewrite it as

$$\Phi_\tau^{j+1} = L\Phi_\tau^{j-1} + 2\tau L_{xx}L_{yy}L_{xy}L_{yx}f^j, \tag{4.4}$$

where

$$
\begin{aligned}
L &= L_{xx}L_{yy}L_{xy}L_{yx}L_{yx}L_{xy}L_{yy}L_{xx}, \qquad (4.5)\\
L_{xx} &= (\theta + (\tau/2)\Lambda_{xx})^{-1}(\theta - (\tau/2)\Lambda_{xx}),\\
L_{yy} &= (\theta + (\tau/2)\Lambda_{yy})^{-1}(\theta - (\tau/2)\Lambda_{yy}),\\
L_{xy} &= (\theta + (\tau/2)\Lambda_{xy})^{-1}(\theta - (\tau/2)\Lambda_{xy}),\\
L_{yx} &= (\theta + (\tau/2)\Lambda_{yx})^{-1}(\theta - (\tau/2)\Lambda_{yx}).
\end{aligned}
$$

Let us investigate the accuracy of scheme (4.4) with respect to time. Substitute the solution of problem (4.1) in (4.4) denoting $\Phi^j = \Phi(t_j)$. Assume then that the time step τ is such that conditions

$$
\begin{aligned}
(\tau/2)\,\|\theta^{-1}\Lambda_{xx}\|_h < 1, &\qquad (\tau/2)\,\|\theta^{-1}\Lambda_{yy}\| < 1,\\
(\tau/2)\,\|\theta^{-1}\Lambda_{xy}\| < 1, &\qquad (\tau/2)\,\|\theta^{-1}\Lambda_{yx}\| < 1
\end{aligned}
\tag{4.6}
$$

are fulfilled. Concrete numerical values of admissible τ are easily obtained on the basis of (3.26), (3.27), (3.28) with the help of (1.6). Then expanding L_{xx}, L_{yy}, L_{xy}, L_{yx} in power τ series, we obtain

$$
\begin{aligned}
\Phi^{j+1} &= [E - 2\tau\theta^{-1}\Lambda - ((2\tau)^2/2)(\theta^{-1}\Lambda)^2]\,\Phi^{j-1}\\
&\quad + 2\tau\theta^{-1}[E + \tau\theta^{-1}\Lambda]f^j + O(\tau^3)
\end{aligned}
\tag{4.7}
$$

or in another representation

$$\theta \frac{\Phi^{j+1} - \Phi^{j-1}}{2\tau} + \Lambda(E - \tau\theta^{-1}\Lambda)\,\Phi^{j-1} = (E - \tau\theta^{-1}\Lambda) f^j. \tag{4.8}$$

In (4.7), (4.8) E is the identity operator.

Excluding the function Φ^{j-1} from the second term in the left-hand side of (4.8) we make use of the expansion

$$\Phi^j = \Phi^{j-1} + (\partial\Phi/\partial t)^{j-1}\,\tau + O(\tau^2). \tag{4.9}$$

For $(\partial\Phi/\partial t)^{j-1}$ the relation

$$(\partial\Phi/\partial t)^{j-1} = -\theta^{-1}\Lambda\Phi^{j-1} + \theta^{-1} f^j + O(\tau) \tag{4.10}$$

is fulfilled with the accuracy $O(\tau)$. Substituting (4.10) in (4.9) we obtain

$$\theta\Phi^j = (\theta - \tau\Lambda)\,\Phi^{j-1} + \tau f^j + O(\tau^2).$$

Hence

$$(\theta - \tau\Lambda)\,\Phi^{j-1} = \theta\Phi^j - \tau f^j + O(\tau^2). \tag{4.11}$$

Substituting (4.11) in (4.8) we arrive at

$$\theta \frac{\Phi^{j+1} - \Phi^{j-1}}{2\tau} + \Lambda\Phi^j = f^j + O(\tau^2).$$

Hence

$$\theta \frac{\partial\Phi}{\partial t} + \Lambda\Phi = f + O(\tau^2), \qquad t = t_j. \tag{4.12}$$

Thus the scheme is of second-order accuracy in time.

Let us analyze the stability of the method. From (4.4) $\|\Phi_\tau^{j+1}\|_h \leqslant \|L\|_h\,\|\Phi^{j-1}\|_h + 2\tau\,\|L_{xx}\|_h\,\|L_{yy}\|_h\,\|L_{xy}\|_h\,\|L_{yx}\|_h\,\|f^j\|_h$. Estimating norms of the operators L_x, L_y, L_{xy}, L_{yx} according to Kellogg's lemma, we have

$$\|L_{xx}\|_h \leqslant 1, \qquad \|L_{yy}\|_h \leqslant 1, \qquad \|L_{xy}\|_h \leqslant 1, \qquad \|L_{yx}\|_h \leqslant 1.$$

Then

$$\|L\|_h \leqslant \|L_{xx}\|_h\,\|L_{yy}\|_h\,\|L_{xy}\|_h\,\|L_{yx}\|_h\,\|L_{yx}\|_h\,\|L_{xy}\|_h \times \|L_{yy}\|_h\,\|L_{xx}\|_h \leqslant 1.$$

Consequently

$$\| \Phi_\tau^{j+1} \|_h \leqslant \| \Phi_\tau^{j-1} \|_h + 2\tau \, \| f^j \|_h,$$

or, making use of the recursion,

$$\| \Phi_\tau^{j+1} \|_h \leqslant c\tau j G, \tag{4.13}$$

where

$$G = \max_j \| f^j \|_h .$$

From (4.13) there follows the stability of (4.3) on the final interval $(0, T]$.

5. Nonuniform Mesh

Expansion of the grid operator of the finite element method in a sum of one-dimensional operators has been obtained for meshes whose nodes are formed by the [illegible] of such an expansion on irregular meshes.

As will be shown, it is possible for irregular meshes of a special type.

Consider Eq. (1.5) for $D \equiv 0$ in the domain $Q = \Omega \times (0, T]$,

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} A \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} B \frac{\partial u}{\partial y} - \frac{\partial}{\partial x} C \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} C \frac{\partial u}{\partial x} = F, \tag{5.1}$$

with the initial conditions

$$u(x, y, 0) = 0 \tag{5.2}$$

and the boundary conditions of two types,

$$u \,|_S = 0, \tag{5.3}$$

$$\left. \frac{\partial u}{\partial \nu} \right|_S = 0, \tag{5.4}$$

where

$$\frac{\partial}{\partial \nu} = \cos(nx) A \frac{\partial}{\partial x} + \cos(ny) B \frac{\partial}{\partial y} + \cos(nx) C \frac{\partial}{\partial y} - \cos(ny) C \frac{\partial}{\partial x}.$$

Conditions (1.6) are fulfilled for the coefficients and the right-hand side. Define weak solutions of problems (5.1)–(5.3) and (5.1), (5.2), (5.4) by the relations

$$\begin{gathered}\tilde{u} \in L_2(0, T; \mathring{W}_2^1(\Omega)), \qquad \frac{\partial \tilde{u}}{\partial t} \in L_2(0, T; \mathring{W}_2^1(\Omega)),\\ \left(\frac{\partial \tilde{u}}{\partial t}, v\right) + I_1(\tilde{u}, v) = (F, v), \qquad v \in \mathring{W}_2^1(\Omega),\\ (\tilde{u}(x, y, 0), v) = 0,\end{gathered} \tag{5.5}$$

and

$$\begin{gathered}\hat{u} \in L_2(0, T; W_2^1(\Omega)), \qquad \frac{\partial \hat{u}}{\partial t} \in L_2(0, T; W_2^1(\Omega)),\\ \left(\frac{\partial \hat{u}}{\partial t}, v\right) + I_1(\hat{u}, v) = (F, v), \qquad v \in W_2^1(\Omega),\\ (\hat{u}(x, y, 0), v) = 0,\end{gathered} \tag{5.6}$$

where

$$I_1(u, v) = \int_\Omega \left(A \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + B \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + C \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - C \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right) dx\, dy.$$

Assume then that the domain Ω is covered with a finite number of quadrangles Π_k satisfying the conditions:

the domain of intersection of the Π_k can be only their boundaries;

the Π_k are convex;

each inner vertex of the quadrangle can be a vertex of only four quadrangles.

Introduce the domain $\tilde{\Omega} \subset \Omega$ with the boundary $\tilde{S}$ as the largest combination of quadrangles belonging to Ω and the domain $\hat{\Omega} \supset \Omega$ with the boundary $\hat{S}$ as the least combination of quadrangles containing $\bar{\Omega}$. Vertices of the quadrangles ordered in some way with respect to the index i will be referred to as mesh nodes. Determine sets of the indices $\tilde{R}^h$, $\tilde{\Gamma}^h$, $\hat{R}^h$, $\hat{\Gamma}^h$ by the relations

$$\begin{aligned}\tilde{R}^h &= \{(i)\colon (x_i, y_i) \in \tilde{\Omega}\}, \qquad & \tilde{\Gamma}^h &= \{(i)\colon (x_i, y_i) \in \tilde{S}\},\\ \hat{R}^h &= \{(i)\colon (x_i, y_i) \in \hat{\Omega}\}, \qquad & \hat{\Gamma}^h &= \{(i)\colon (x_i, y_i) \in \hat{S}\}.\end{aligned}$$

Now, divide each of the quadrangles Π_k by one of the diagonals into the two triangles T_k^1, T_k^2 and assume that the areas of the domains $\Omega \backslash T_k^e$ are not less than the value sh_0^2 for any triangles T_k^e. Here and later h_0 is the biggest of the sides of the triangles in $\hat{\Omega}$ and s is constant. Determine the functions $\omega_i(x, y)$ for $i \in \hat{R}^h \cup \hat{\Gamma}^h$ which are continuous in Ω linear on each triangle $T_k^e \subset \hat{\Omega}$, such that

$$\begin{aligned} \omega_i(x_j, y_j) &= 1, \qquad i = j, \\ &= 0, \qquad i \neq j, \end{aligned} \qquad i, j \in \hat{R}^h \cup \hat{\Gamma}^h.$$

Introduce also the functions

$$\tilde{\varphi}(x, y, t) = \sum_{i \in \tilde{R}^h} \varphi_i(t)\, \omega_i(x, y), \tag{5.7}$$

$$\hat{\varphi}(x, y, t) = \sum_{i \in \hat{R}^h \cup \hat{\Gamma}^h} \varphi_i(t)\, \omega_i(x, y) \tag{5.8}$$

to be called approximate solutions of problems (5.5), (5.6), respectively, if they satisfy the relations

$$\begin{aligned} &\left(\frac{\partial \tilde{\varphi}}{\partial t}, \omega_i\right) + I_1(\tilde{\varphi}, \omega_i) = (F, \omega_i), \qquad i \in \tilde{R}^h, \\ &(\tilde{\varphi}(x, y, 0), \omega_i) = 0, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &\left(\frac{\partial \hat{\varphi}}{\partial t}, \omega_i\right) + I_1(\hat{\varphi}, \omega_i) = (F, \omega_i), \qquad i \in \hat{R}^h \cup \hat{\Gamma}^h, \\ &(\hat{\varphi}(x, y, 0), \omega_i) = 0. \end{aligned} \tag{5.10}$$

Estimates for differences of the solutions $\tilde{\zeta} = \tilde{u} - \tilde{\varphi}$, $\hat{\zeta} = \hat{u} - \hat{\varphi}$ are also obtained, as in the case of regular meshes on the basis of the approximation theorems for irregular meshes, proved in [8]. They are of the form

$$\|\tilde{\zeta}\|_{L_\infty(0,T;L_2(\Omega))} + \|\tilde{\zeta}\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} \leqslant c_1 h_0, \tag{5.11}$$

$$\|\hat{\zeta}\|_{L_\infty(0,T;L_2(\Omega))} + \|\hat{\zeta}\|_{L_2(0,T;\mathring{W}_2^1(\Omega))} \leqslant c_2 h_0. \tag{5.12}$$

The improvement of estimates (5.11) to the order h_0 as compared to similar estimates (2.15), (2.16) for regular meshes is due to the use of irregular grid allowing approximation of the boundary conditions (5.3) to the degree of h_0 [8].

Let us analyze the properties of the grid operators determined by relations (5.9), (5.10). It is difficult to carry out the analysis in the coordinates (x, y). For simplicity, let us make use of isoparametric transformations [9].

Introduce a piecewise-linear transformation $\mathscr{L}: (x, y) \to (\xi, \eta)$ satisfying the conditions:

(a) for each triangle T_k^e the transformation $\mathscr{L}$ is linear;

(b) $\mathscr{L}$ transforms each quadrangle Π_k into a unit square P_k;

(c) $\mathscr{L}$ transforms the domain Ω into the domain Δ, $\tilde{\Omega}$ into $\tilde{\Delta}$, and $\hat{\Omega}$ into $\hat{\Delta}$ so

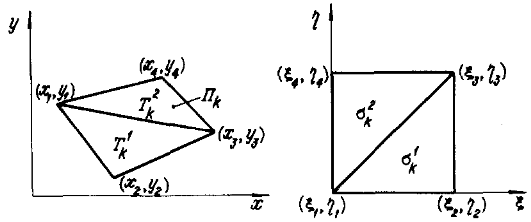

FIG. 2.

that topology of $\tilde{\Omega}$ and $\hat{\Omega}$ is preserved; that is, the ordering of the squares P_k and the common sides and the vertices of Π_k remain common sides and vertices of P_k.

If in each quadrangle Π_k the vertices are counterclockwise ordered so that index (1) denotes a vertex that is nearest to the origin in the plane (ξ, η), it is easy to see that the form of transformation $\mathscr{L}$ depends on the kinds of partitioning the quadrangle Π_k into triangles by the diagonal.

Let us consider both cases of partitioning:

(i) The diagonal connects points (x_1, y_1) and (x_3, y_3) as shown in Fig. 2. If we take $\xi_3 = \xi_2 = \xi_1 + 1 = \xi_4 + 1$, $\eta_3 = \eta_4 = \eta_1 + 1 = \eta_2 + 1$ into account, transformation is of the form

$$\text{for } T_k^1: \quad x = x_1 + (x_2 - x_1)(\xi - \xi_1) + (x_3 - x_2)(\eta - \eta_1),$$
$$y = y_1 + (y_2 - y_1)(\xi - \xi_1) + (y_3 - y_2)(\eta - \eta_1), \tag{5.13}$$

$$\text{for } T_k^1: \quad x = x_1 + (x_3 - x_4)(\xi - \xi_1) + (x_4 - x_1)(\eta - \eta_1),$$
$$y = y_1 + (y_3 - y_4)(\xi - \xi_1) + (y_4 - y_1)(\eta - \eta_1). \tag{5.14}$$

(ii) The quadrangle is triangulated by the diagonal connecting points (x_2, y_2) and (x_4, y_4) (Fig. 3). Then the transformation is of the form

$$\begin{aligned} \text{for } T_k^1: \quad x &= x_1 - (x_1 - x_2)(\xi - \xi_1) + (x_4 - x_1)(\eta - \eta_1), \\ y &= y_1 - (y_1 - y_2)(\xi - \xi_1) + (y_4 - y_1)(\eta - \eta_1), \end{aligned} \tag{5.15}$$

$$\begin{aligned} \text{for } T_k^2: \quad x &= x_2 - (x_4 - x_3)(\xi - (\xi_1 + 1)) + (x_3 - x_2)(\eta - \eta_1), \\ y &= y_2 - (y_4 - y_3)(\xi - (\xi_1 + 1)) + (y_3 - y_2)(\eta - \eta_1). \end{aligned} \tag{5.16}$$

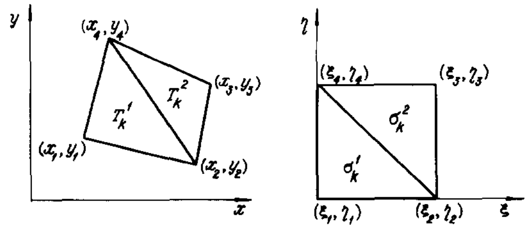

FIG. 3.

The Galerkin equation (5.9), (5.10) in the coordinates (ξ, η) can be rewritten

$$\left(\frac{\partial \tilde{\psi}}{\partial t}, w_i\right)_1 + I(\tilde{\psi}, w_i) = (f, w_i)_1, \qquad i \in \tilde{R}^h, \tag{5.17}$$
$$(\tilde{\psi}(\xi, \eta, 0), w_i)_1 = 0.$$

$$\left(\frac{\partial \hat{\psi}}{\partial t}, w_i\right)_1 + I(\hat{\psi}, w_i) = (f, w_i)_1, \qquad i \in \hat{R}^h \cup \hat{\Gamma}^h,$$
$$(\hat{\psi}(\xi, \eta, 0), w_i)_1 = 0. \tag{5.18}$$

Here we use the notation

$$I(u, v) = \int_\Delta \left(\tilde{A}\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \xi} + \tilde{B}\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \eta} + \tilde{D}\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} + \tilde{D}\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} + \tilde{C}\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} - \tilde{C}\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}\right) d\xi\, d\eta,$$

$$(u, v)_1 = \int_\Delta uv\, |J|\, d\xi\, d\eta, \quad .$$

$$w_i(\xi, \eta) = \omega_i(x(\xi, \eta), y(\xi, \eta)),$$
$$\tilde{\psi}(\xi, \eta, t) = \tilde{\varphi}(x(\xi, \eta), y(\xi, \eta), t) = \sum_{i \in \tilde{R}^h} \varphi_i(t)\, w_i(\xi, \eta), \tag{5.19}$$
$$\hat{\psi}(\xi, \eta, t) = \hat{\varphi}(x(\xi, \eta), y(\xi, \eta), t) = \sum_{i \in \hat{R}^h \cup \hat{\Gamma}^h} \varphi_i(t)\, w_i(\xi, \eta),$$
$$f(\xi, \eta, t) = F(x(\xi, \eta), y(\xi, \eta), t),$$

where transformation formulas $x(\xi, \eta)$, $y(\xi, \eta)$ are given on each triangle T_k^e by one of the relations (5.13)–(5.16) depending on the triangulation of the quadrangle Π_k, where T_k^e belongs. Depending on the form of transformation, the Jacobian $|J|$ and the coefficients $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are as follows: in the case of triangulation of form (i),

for T_k^1:

$$|J| = |(x_2 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_2 - y_1)|,$$
$$\tilde{A} = A_1 = \frac{1}{|J|}\,[a(y_3 - y_2)^2 + b(x_3 - x_2)^2],$$
$$\tilde{B} = B_1 = \frac{1}{|J|}\,[a(y_2 - y_1)^2 + b(x_2 - x_1)^2], \tag{5.20}$$
$$\tilde{D} = D_1 = \frac{1}{|J|}\,[[a(y_3 - y_2)(y_2 - y_1) + b(x_3 - x_2)(x_2 - x_1)],$$
$$\tilde{C} = C_1 = c,$$

and for T_k^2:

$$|J| = |(x_3 - x_4)(y_4 - y_1) - (x_4 - x_1)(y_3 - y_4),$$

$$\tilde{A} = A_2 = \frac{1}{|J|}\,[a(y_4 - y_1)^2 + b(x_4 - x_1)^2], \tag{5.21}$$

$$\tilde{B} = B_2 = \frac{1}{|J|}\,[a(y_3 - y_4)^2 + b(x_3 - x_4)^2],$$

$$\tilde{D} = D_2 = \frac{1}{|J|}\,[a(y_3 - y_4)(y_4 - y_1) + b(x_3 - x_4)(x_4 - x_1)],$$

$$\tilde{C} = C_2 = c,$$

or in the case of triangulation of the type (ii),

for T_k^1:

$$|J| = |(x_4 - x_1)(y_1 - y_2) - (y_4 - y_1)(x_1 - x_2)|$$

$$\tilde{A} = A_1 = \frac{1}{|J|}\,[a(y_4 - y_1)^2 + b(x_4 - x_1)^2],$$

$$\tilde{B} = B_1 = \frac{1}{|J|}\,[a(y_1 - y_2)^2 + b(x_1 - x_2)^2], \tag{5.22}$$

$$\tilde{D} = D_1 = \frac{1}{|J|}\,[a(y_4 - y_1)(y_1 - y_2) + b(x_4 - x_1)(x_1 - x_2)],$$

$$\tilde{C} = C_1 = c$$

and for T_k^2:

$$|J| = |(x_3 - x_2)(y_4 - y_3) - (y_3 - y_2)(x_4 - x_3)|,$$

$$\tilde{A} = A_2 = \frac{1}{|J|}\,[a(y_3 - y_2)^2 + b(x_3 - x_2)^2],$$

$$\tilde{B} = B_2 = \frac{1}{|J|}\,[a(y_4 - y_3)^2 + b(x_4 - x_3)^2], \tag{5.23}$$

$$\tilde{D} = D_2 = \frac{1}{|J|}\,[a(y_3 - y_2)(y_4 - y_3) + b(x_3 - x_2)(x_4 - x_3)],$$

$$\tilde{C} = C_2 = c.$$

In (5.20)–(5.23) we use the notation

$$a = A(x(\xi, \eta), y(\xi, \eta)),$$
$$b = B(x(\xi, \eta), y(\xi, \eta)),$$
$$c = C(x(\xi, \eta), y(\xi, \eta)).$$

It should be noted that since the quadrangles Π_k are convex, the values of their angles are bounded within $0 < \delta_0 \leqslant \alpha \leqslant \delta_1 < \pi$ where $\delta_0, \delta_1 > 0$ are constants, consequently the Jacobian $|J|$ of (5.13)–(5.16) does not turn into zero anywhere in Ω [18]. Here, the coefficients $A, B, C \in L_2(\Omega)$ and thus $\tilde{A}, \tilde{B}, \tilde{D}, \tilde{C} \in L_2(\Delta)$. In particular, if A, B, C are approximated by piecewise-linear functions on the triangulation domain Ω the coefficients $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ remain linear on each triangle $\sigma_k^e \subset \hat{\Delta}$ and can have only finite discontinuities in the vertices and along the sides of the triangles.

Besides, let us note that the definition of transformation $\mathscr{L}$ implies that in $\hat{\Delta} = \bigcup_k P_k$ mesh nodes are formed by intersection of the lines parallel to the axes $0\xi, 0\eta$ and are apart from each other at distance unity $\xi_m = \xi_0 + m$, $\eta_n = \eta_0 + n$. Here m and n are the numbers of the lines with respect to ξ and η correspondingly. If the origin is selected so that $\xi_0 = \eta_0 = 0$ then m and n are coordinates of the mesh nodes with respect to ξ and η correspondingly.

In $\hat{\Omega}$, mesh nodes were i-index-ordered. In the transformed domain Δ it is convenient to change the notation of the mesh points making use of indices–coordinates of the nodes (m, n) $(m = \xi_i,\ n = \eta_i)$. Then relations (5.19) become

$$\begin{aligned} \tilde{\psi}(\xi, \eta, t) &= \sum_{(m,n) \in \tilde{\Delta}} \psi_{m,n}(t)\, w_{m,n}(\xi, \eta), \\ \hat{\psi}(\xi, \eta, t) &= \sum_{(m,n) \in \hat{\Delta} \cup \hat{S}} \psi_{m,n}(t)\, w_{m,n}(\xi, \eta), \\ \psi_{m,n}(t) &= \varphi_i(t), \qquad w_{m,n}(\xi, \eta) = w_i(\xi, \eta), \qquad (\xi_i, \eta_i) = (m, n). \end{aligned} \tag{5.24}$$

The $w_{m,n}$ are functions which are continuous in Δ linear on each triangle, such that

$$\begin{aligned} w_{m,n}(k, e) &= 1, \qquad (m, n) = (k, e), \\ &= 0, \qquad (m, n) \neq (k, e). \end{aligned} \tag{5.25}$$

Having applied the "lumping" method to calculation of the mass matrix we can write (5.17), (5.18) as

$$\frac{\partial \psi_{m,n}}{\partial t} \int_\Delta w_{m,n} |J|\, d\xi\, d\eta + I(\tilde{\psi}, w_{m,n}) = (f, w_{m,n}), \qquad (m, n) \in \tilde{\Delta}, \tag{5.26}$$

$$\psi_{m,n}(0) = 0,$$

$$\frac{\partial \psi_{m,n}}{\partial t} \int_\Delta w_{m,n} |J|\, d\xi\, d\eta + I(\hat{\psi}, w_{m,n}) = (f, w_{m,n}), \qquad (m, n) \in \hat{\Delta} \cup \hat{S}, \tag{5.27}$$

$$\psi_{m,n}(0) = 0.$$

We note that problems (5.26), (5.27) coincide with (3.1), (3.2), notation inclusive; that is, they reduce to the form (3.3) with the coefficients of the operator Λ determined by relations (3.5) accurate to within the notation and the fact that in this case

$\partial w_{m,n}/\partial\xi$, $\partial w_{m,n}/\partial\eta$ equal $1, 0$, or -1 and $h=k=1$. Then the grid operator Λ, approximating the differential space operator, can be described as a sum of the four one-dimensional operators

$$\Lambda = \Lambda_{\xi\xi} + \Lambda_{\eta\eta} + \Lambda_{\xi\eta} + \Lambda_{\eta\xi}. \tag{5.28}$$

For scheme (4.3) to be stable, the operators $\Lambda_{\xi\xi}$, $\Lambda_{\eta\eta}$, $\Lambda_{\xi\eta}, \Lambda_{\eta\xi}$ should be positive semi-definite. Positive definiteness of the operator θ at the time derivative is obvious. The condition of positive semidefiniteness of the operators $\Lambda_{\xi\xi}$, $\Lambda_{\eta\eta}$, $\Lambda_{\xi\eta}$, $\Lambda_{\eta\xi}$ is obtained in a similar way as relations (3.15), (3.17), (3.20), (3.23), respectively.

In triangulation of the quadrangles of form (i) these conditions can be written

$$\begin{gathered}
\int_{\sigma_k^1\cap\Delta} A_1\, d\xi\, d\eta \geqslant \int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta, \\
\int_{\sigma_k^1\cap\Delta} B_1\, d\xi\, d\eta \geqslant \int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta, \\
\int_{\sigma_k^2\cap\Delta} A_2\, d\xi\, d\eta \geqslant \int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta, \\
\int_{\sigma_k^2\cap\Delta} B_2\, d\xi\, d\eta \geqslant \int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta, \\
\int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta + \int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta \geqslant 0.
\end{gathered} \tag{5.29}$$

For triangulation of the form (ii) for transformations (5.15), (5.16) conditions can be represented as

$$\begin{gathered}
\int_{\sigma_k^1\cap\Delta} A_1\, d\xi\, d\eta \geqslant -\int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta, \\
\int_{\sigma_k^1\cap\Delta} B_1\, d\xi\, d\eta \geqslant -\int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta, \\
\int_{\sigma_k^2\cap\Delta} A_2\, d\xi\, d\eta \geqslant -\int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta, \\
\int_{\sigma_k^2\cap\Delta} B_2\, d\xi\, d\eta \geqslant -\int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta, \\
\int_{\sigma_k^1\cap\Delta} D_1\, d\xi\, d\eta + \int_{\sigma_k^2\cap\Delta} D_2\, d\xi\, d\eta \leqslant 0.
\end{gathered} \tag{5.30}$$

Substituting the expressions for coefficients $A_1, B_1, D_1, A_2, B_2, D_2$ in (5.29), after algebraic transformations, and changing to the coordinates (x, y) yields

$$\begin{aligned}
&\bar{A}_1[(y_3-y_2)^2 + (y_3-y_1)^2 - (y_2-y_1)^2] \\
&\quad + \bar{B}_1[(x_3-x_2)^2 + (x_3-x_1)^2 - (x_2-x_1)^2] \geqslant 0,
\end{aligned} \tag{5.31}$$

$$\bar{A}_1[(y_2-y_1)^2+(y_3-y_1)^2-(y_3-y_2)^2] \\ +\bar{B}_1[(x_2-x_1)^2+(x_3-x_1)^2-(x_3-x_2)^2]\geqslant 0,$$

$$\bar{A}_2[(y_4-y_1)^2+(y_3-y_1)^2-(y_4-y_3)^2] \\ +\bar{B}_2[(x_4-x_1)^2+(x_3-x_1)^2-(x_4-x_3)^2]\geqslant 0,$$

$$\bar{A}_2[(y_4-y_3)^2+(y_3-y_1)^2-(y_4-y_1)^2] \\ +\bar{B}_2[(x_4-x_3)^2+(x_3-x_1)^2-(x_4-x_1)^2]\geqslant 0,$$

$$\bar{A}_2[(y_4-y_3)^2+(y_4-y_1)^2-(y_3-y_1)^2] \\ +\bar{B}_2[(x_4-x_3)^2+(x_4-x_1)^2-(x_3-x_1)^2] \\ +\bar{A}_1[(y_3-y_2)^2+(y_2-y_1)^2-(y_3-y_1)^2] \\ +\bar{B}_1[(x_3-x_2)^2+(x_2-x_1)^2-(x_3-x_1)^2]\geqslant 0.$$

Here

$$\bar{A}_1=\int_{T_k^1\cap\Omega} A\,dx\,dy, \qquad \bar{B}_1=\int_{T_k^1\cap\Omega} B\,dx\,dy,$$

$$\bar{A}_2=\int_{T_k^2\cap\Omega} A\,dx\,dy, \qquad \bar{B}_2=\int_{T_k^2\cap\Omega} B\,dx\,dy.$$

Relations (5.31), as indicated earlier, must be fulfilled for all $T_k^e\subset\Omega_1$. To calculate conditions (5.30), we substitute the values A_1, B_1, D_1, A_2, B_2, D_2 making use of (5.22), (5.23). As a result we arrive at the inequalities

$$\begin{aligned}
&\bar{A}_1[(y_4-y_1)^2+(y_4-y_2)^2-(y_2-y_1)^2] \\
&\quad+\bar{B}_1[(x_4-x_1)^2+(x_4-x_2)^2-(x_2-x_1)^2]\geqslant 0, \\
&\bar{A}_1[(y_2-y_1)^2+(y_4-y_2)-(y_4-y_1)^2] \\
&\quad+\bar{B}_1[(x_2-x_1)^2+(x_4-x_2)-(x_4-x_1)^2]\geqslant 0, \\
&\bar{A}_2[(y_3-y_2)^2+(y_4-y_2)^2-(y_4-y_3)^2] \\
&\quad+\bar{B}_2[(x_3-x_2)^2+(x_4-x_2)^2-(x_4-x_3)^2]\geqslant 0, \\
&\bar{A}_2[(y_4-y_3)^2+(y_4-y_2)^2-(y_3-y_2)^2] \\
&\quad+\bar{B}_2[(x_4-x_3)^2+(x_4-x_2)^2-(x_3-x_2)^2]\geqslant 0, \\
&\bar{A}_1[(y_4-y_1)^2+(y_2-y_1)^2-(y_4-y_2)^2] \\
&\quad+\bar{B}_1[(x_4-x_1)^2+(x_2-x_1)^2-(x_4-x_2)^2] \\
&\quad+\bar{A}_2[(y_3-y_2)^2+(y_4-y_3)^2-(y_4-y_2)^2] \\
&\quad+\bar{B}_2[(x_3-x_2)^2+(x_4-x_3)^2-(x_4-x_2)^2]\geqslant 0.
\end{aligned} \tag{5.32}$$

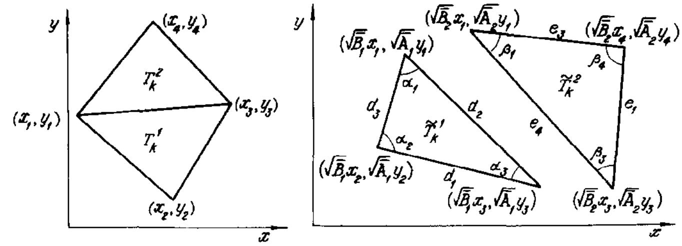

FIG. 4.

Conditions (5.31), (5.32) have a clear geometric sense. Analyze, for example, relations (5.31). Let the triangle T_k^1 be deformed along the axes x and y proportionallly to $\sqrt{B_1}$ and $\sqrt{A_1}$, respectively, and the triangle T_k^2 be deformed along the axes x and y proportionally to $\sqrt{B_2}$ and $\sqrt{A_2}$, respectively (Fig. 4). If we denote the lengths of the sides of the resulting triangles $\tilde{T}_k^2$ by d_1, d_2, d_3 and e_1, e_4, e_3 and their angles by $\alpha_1, \alpha_2. \alpha_3$ and $\beta_1, \beta_2, \beta_3$, respectively, conditions (5.31) will be written

$$\begin{gathered} d_1^2 + d_2^2 - d_3^2 \geqslant 0, \qquad d_3^2 + d_2^2 - d_1^2 \geqslant 0, \\ e_3^2 + e_4^2 - e_1^2 \geqslant 0, \qquad e_1^2 + e_4^2 - e_3^2 \geqslant 0, \\ d_3^2 + d_1^2 - d_2^2 + e_1^2 + e_3^2 - e_4^2 \geqslant 0. \end{gathered} \tag{5.33}$$

Rembering that

$$d_1^2 + d_2^2 - d_3^2 = 2d_2 d_1 \cos \alpha_3, \ldots, \text{etc.},$$

we rewrite (5.31) as the conditions for $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$,

$$\begin{gathered} \alpha_3 \leqslant \pi/2, \qquad \alpha_1 \leqslant \pi/2, \qquad \beta_1 \leqslant \pi/2, \qquad \beta_3 \leqslant \pi/2, \\ d_1 d_3 \cos \alpha_2 + e_1 e_3 \cos \beta_4 \geqslant 0. \end{gathered} \tag{5.34}$$

Thus, the adjacent angles in the triangles T_k^1, T_k^2 may not be greater than the right angle and one of the alternate angles may be greater than $\pi/2$ while the other may be as much less than $\pi/2$ as to satisfy the last of relations in (5.33).

Of a similar sense are conditions (5.32); however, in this case the alternate angles are those at the vertices (x_2, y_2) and (x_4, y_4).

The alternate angles must meet the conditions similar to the last one from (5.33). If Π_k is such that it is impossible to fulfill these conditions, the mesh must be rearranged.

6. Determination of the Time Step

The convergence of scheme (4.3) was proved under the assumption that $\tau = \tau_0$, where τ_0 is determined by conditions (4.6) and has values of order $\tau \simeq O(\min(\Delta x_{\min}^2,$

$\Delta y^2_{\min}$)). This is a severe constraint on the time interval that brings its value nearer to the time interval for the explicit scheme. The question arises as to what practically satisfactory values of τ can be chosen in the suggested scheme for the given space interval. We shall try to answer this question by carrying out numerical calculations for a concrete equation which possesses sufficiently "bad" properties: discontinuity coefficients and a solution with the boundary layer. Let us consider the equation

$$E\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}A\frac{\partial u}{\partial x}-\frac{\partial}{\partial y}B\frac{\partial u}{\partial y}-\frac{\partial}{\partial x}C\frac{\partial u}{\partial y}+\frac{\partial}{\partial y}C\frac{\partial u}{\partial x}=f \tag{6.1}$$

in the domain Ω bounded by the curves

$$x=0, \quad x=1, \quad y=0, \quad y=H(x)\equiv 1-(1/\pi)\sin \pi x \tag{6.2}$$

under the conditions

$$u\,|_S=0. \tag{6.3}$$

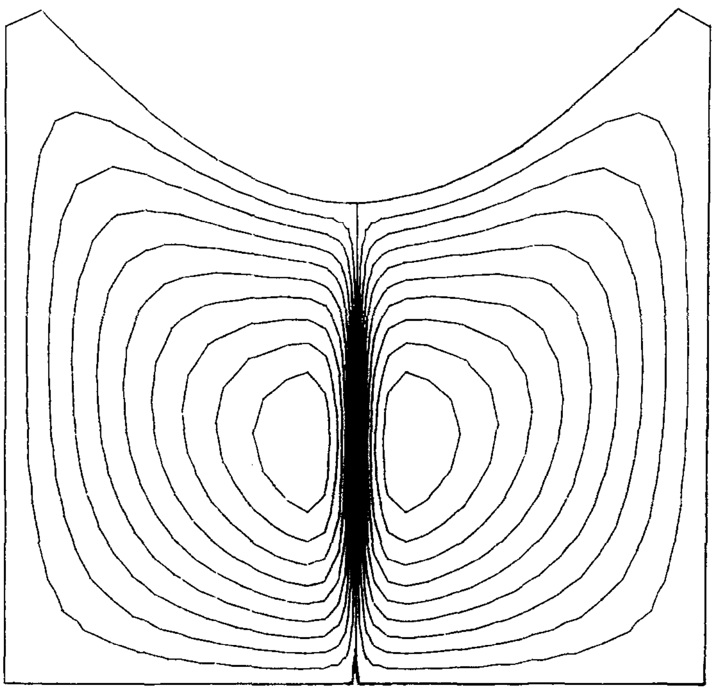

FIG. 5. Exact solution, $t=0.1$.

The coefficients of the equation have the form

$$E = A = H^{-1}, \qquad B = \gamma H - y^2 \cos^2 \pi x \cdot H^{-3},$$
$$\gamma = 10, \qquad C = -\beta y H^{-1},$$
$$\beta = -50, \qquad 0 \leqslant x < \tfrac{1}{2}.$$
$$= 0, \qquad x = \tfrac{1}{2},$$
$$= 50, \qquad \tfrac{1}{2} < x \leqslant 1.$$

The right-hand side of Eq. (6.1), f, is chosen in the form corresponding to the solution

$$\begin{aligned} u &= \frac{\alpha}{\gamma\pi} e^{-\pi^2 t} \sin\left(\frac{\pi y}{1 - (1/\pi)\sin \pi x}\right) \varphi(x), \\ \varphi(x) &= p e^{a_1 x} + q e^{a_2 x} - 1, \qquad 0 \leqslant x < \tfrac{1}{2}, \\ &= p e^{a_1 (x - (1/2))} + q e^{a_2 (x - (1/2))} - 1, \qquad \tfrac{1}{2} \leqslant x \leqslant 1, \\ q &= \frac{e^{a_1/2} - 1}{e^{a_1/2} - e^{a_2/2}}, \qquad p = 1 - q, \\ a_1 &= -(\beta/2) + \sqrt{(\beta^2/4) + \pi^2\gamma}, \qquad a_2 = -(\beta/2) - \sqrt{(\beta^2/4) + \pi^2\gamma}, \\ \alpha &= \ \ 1, \qquad 0 \leqslant x < \tfrac{1}{2}, \\ &= 0, \qquad x = \tfrac{1}{2}, \\ &= -1, \qquad \tfrac{1}{2} \leqslant x \leqslant 1. \end{aligned} \tag{6.4}$$

The form of the piecewise-linear representation of function (6.4) at the moment $t = 10^{-1}$ is shown in Fig. 5. The function u posseses the boundary value in the neighbourhood of the line $x = \frac{1}{2}$. Having in mind the form of the solution and the geometry of the domain Ω, we chose an irregular grid. It consists of the points

$$x_i = x_{i-1} + \Delta x_i \qquad (\max \Delta x_i = 0.05, \qquad \min \Delta x_i = 0.005),$$
$$y_{i,j} = z_j(1 - (1/\pi) \sin \pi x_i), \qquad z_j = z_{j-1} + \Delta z, \quad \Delta z = 1/14.$$

The meshes are triangulated as shown in Fig. 6. Calculations were carried out for $\tau = 2 \times 10^{-3}$, 10^{-3}, 2×10^{-4}, 10^{-4}, 2×10^{-5}. In all the variants of the calculation the absolute error maximum is concentrated at the line of discontinuity of the coefficients, $x = \frac{1}{2}$.

Figure 7 shows an approximate solution at $t = 0.1$, $\tau = 2 \times 10^{-4}$.

As the parameter τ increases, the errors of the solution near the line $x = \frac{1}{2}$ increase. Figure 8 describes approximate solutions with a change of τ on the line $y = \frac{1}{2}(1 - (1/\pi) \sin \pi x)$. As the picture is antisymmetric with respect to $x = \frac{1}{2}$, the figure shows only one half of the graph of the function. One can see that the solution is described fairly well when $\tau = 10^{-4}$.

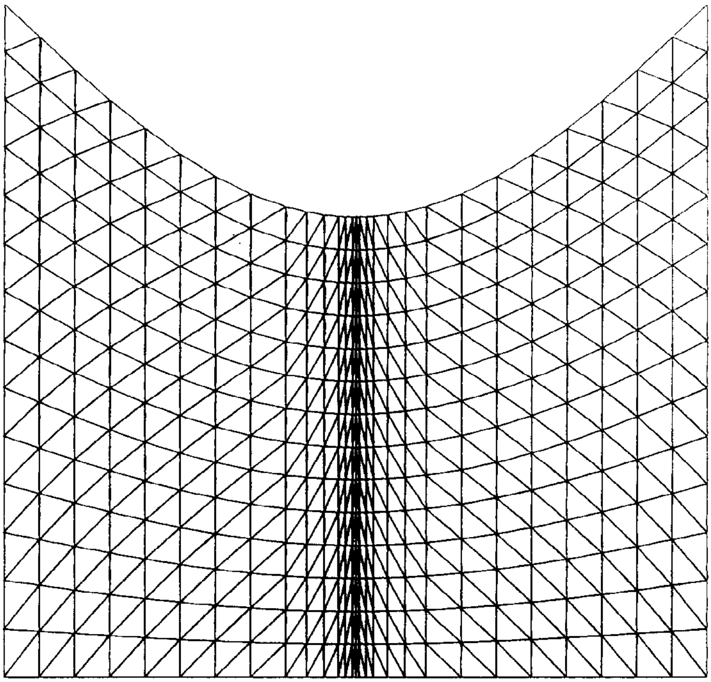

FIG. 6. Grid domain.

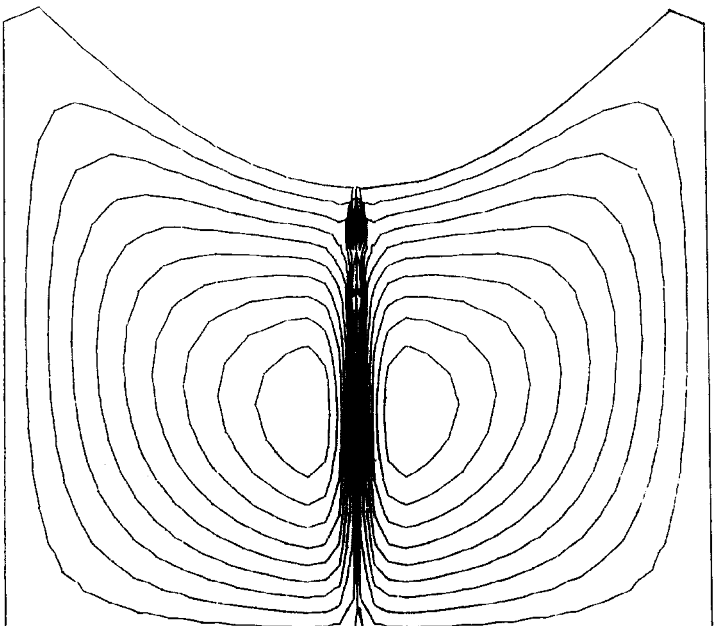

FIG. 7. Numerical solution, $\tau = 2 \times 10^{-4}$, $t = 0.1$.

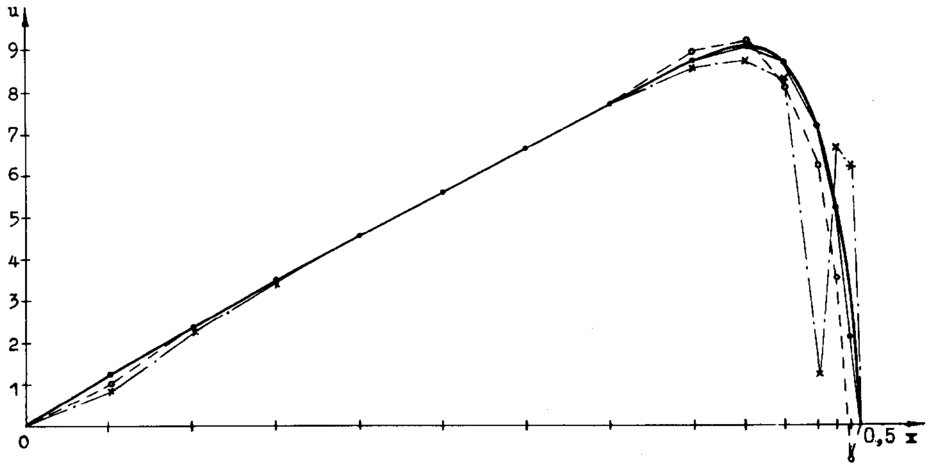

FIG. 8. —, exact solution; —·—, $\tau = 10^{-4}$; —○—, $\tau = 10^{-3}$; · —*— ·, $\tau = 2 \times 10^{-3}$.

Relative deviation σ of the approximate solution from the exact one was calculated by formula $\sigma = \|\bar{u} - u\|_h / \|u\|_h$. The calculations show that for $\tau = 2 \times 10^{-3}$, $\tau = 10^{-3}$, $\tau = 2 \times 10^{-4}$, $\tau = 10^{-4}$ the value of σ remains constant with time. For $\tau = 2 \times 10^{-5}$ the value of σ increases with time in the first steps. Then the error approaches an asymptotic value which does not vary with time. This is due to the fact that in this case in the first time steps there is an influence of the initial conditions which are exact. As one moves away from the initial conditions their influence dies down and the error assumes a value characterizing the accuracy of the scheme. The values of σ are shown in Table I.

7. Applications of the Method

(i) It has been indicated that the proposed algorithm has been used in the program designed to calculate optimal distribution of pollution sources in a water basin as a barotropic approximation [13]. The equations of the pollution concentration φ are of the form

$$\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} \psi \frac{\partial \varphi}{\partial y} - \frac{\partial}{\partial y} \psi \frac{\partial \varphi}{\partial x} - \mu \Delta \varphi = Q\, \delta(x - x_0, y - y_0). \tag{7.1}$$

TABLE I

τ:	2×10^{-3}	10^{-3}	2×10^{-4}	10^{-4}	2×10^{-5}
σ:	0.752	0.593	0.261	0.172	0.127

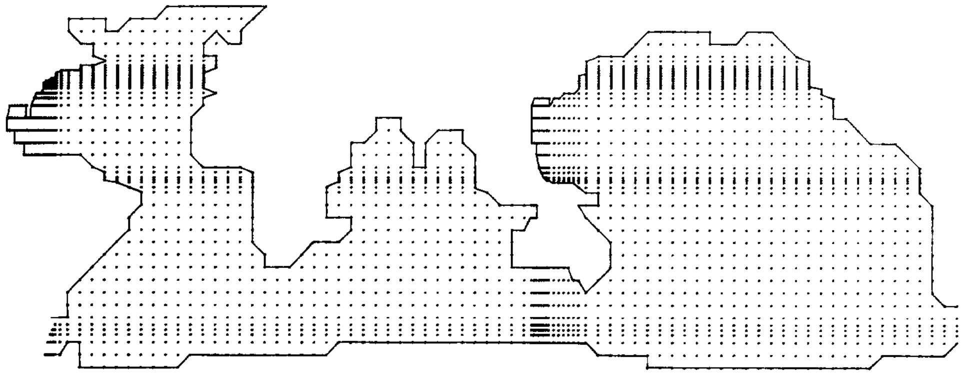

FIG. 9. Grid domain for the solution of problem (7.2), (7.3).

Here ψ is the stream function, and Q is the pollution source concentration in the point (x_0, y_0).

(ii) Based on the method described in the paper, we have developed the algorithm and carried out calculations to determine the integral stream function in the domain approximating the world ocean in a nonlinear formulation [22]. The equations for the vorticity ζ and the stream function ψ written in the spherical coordinate system are of the form

$$\left(\frac{\partial}{\partial t}+R\right)\zeta+J\left[\frac{\zeta}{H},\psi\right]-\nu\left(\Delta\zeta-\frac{2}{a^2}\zeta\right)=F. \tag{7.2}$$

$$\Delta_H\psi=\zeta-f, \tag{7.3}$$

Here $\Delta q=\nabla\cdot(\nabla q)$, $\Delta_H q=\nabla\cdot(1/H\,\nabla q)$, $J[p,q]=(p_\lambda q_\theta-p_\theta q_\lambda)/a^2\cos\theta$, λ, θ are the spherical coordinates, ∇ is the operator of gradient on a sphere of radius a, a is

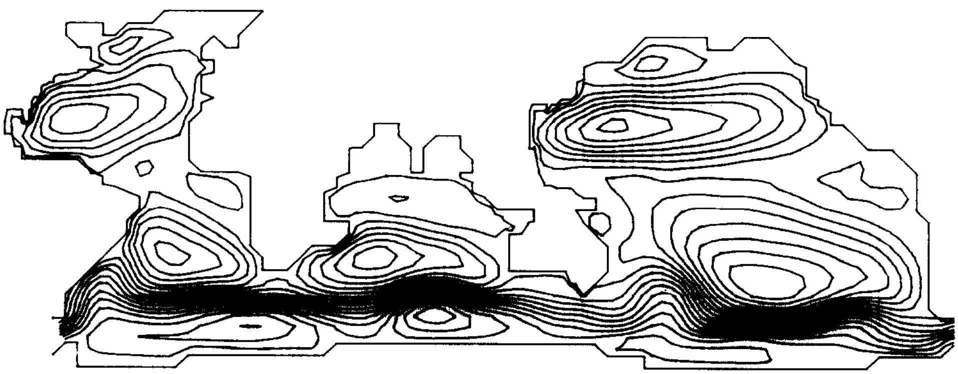

FIG. 10. An example of the stream function pattern calculated using the finite element and splitting-up methods.

the earth's radius, R, ν are the friction parameters, f is the Coriolis parameter, $H(\lambda, \theta)$ is the function of the bottom relief. To solve the problem a mesh has been chosen which is thickening in the regions of the intensive currents (Fig. 9). An example of the stream function calculated is presented in Fig. 10.

(iii) The algorithm has been used in a baroclinic model of the ocean global circulation [19]. In that case the splitting-up scheme has been applied to solve two-dimensional equations resulting from the splitting of the three-dimensional equation of heat diffusion in the ocean. At each step of the splitting the two-dimensional equations are of the form

$$\frac{\partial T}{\partial t} + \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial T}{\partial x_j} = 0, \qquad a_{ij} \neq a_{ji}. \tag{7.4}$$

These equations coincide with Eq. (1.1) to within the notations and they are solved with the help of the algorithm described above.

Conclusion

For the parabolic equations with mixed derivatives of the form (1.5), under the boundary conditions (1.3), (1.4), space approximations have been built by the Galerkin method on regular meshes using piecewise-linear trial functions. The difference operators obtained can be split into four difference operators applied in coordinate and diagonal directions. A condition of positive semidefiniteness of one-dimensional operators has been obtained whih imposes stronger constraints on the coefficients of the equation than the conditions of uniform ellipticity of the space operator of the problem (1.6*). This is natural because in this particular case the correctness must exist also for the split one-dimensional problems. It can be shown that the splitting in a differential form of Eq. (1.5) results in the conditions similar to (3.15), (3.17).

Positive semidefiniteness of one-dimensional difference operators can be used to substantiate the two-cycle splitting-up method which has convergence with respect to time $O(\tau_2)$ for $\tau < \tau_0$. Numerical calculations show that the values τ can be chosen significantly larger than τ_0 even for problems with discontinuous coefficients.

The authors have considered and substantiated a possibility to carry out the splitting of the difference operator of the Galerkin method on an irregular grid for the case when the coefficient D at mixed derivatives is equal to zero. The stability conditions obtained reduce to geometrical conditions on a deformed mesh.

The examples presented in the last section show that the algorithm may be useful in some physical application.

Acknowledgment

The authors wish to acknowledge that the figures presented in the paper were drawn using procedures of INTERLIB [23] and SMOG libraries.

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